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Examination of Food Security Dynamics in the Presence of Measurement Error*

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Abstract

We examine intra- and intergenerational food security dynamics in the United States using longitudinal data from the Panel Study of Income Dynamics (PSID) while accounting for measurement error. To proceed, we apply recently developed methods on the partial identification of transition matrices. We show that accounting for measurement error is crucial as even modest errors can dwarf the information contained in the data. Nonetheless, we find that much can be learned under fairly weak assumptions; the strongest and most informative being that measurement errors are serially uncorrelated. In particular, while the evidence – both intragenerational and intergenerational – is consistent with significant mobility, we also find food security status to be persistent for at least some households in the tails of the distribution. Finally, we document some heterogeneities in the dynamics across households differentiated by race an education.

JEL: C18, D31, I32

Keywords: Food Security, Mobility, Measurement Error, Partial Identification, Poverty

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Executive Summary

Food insecurity continues to be one of the most significant public health concerns facing the United States (Gundersen et al. 2011). Recent pre-pandemic data from 2018 suggests that 11.1% of U.S. households were food insecure meaning these households lacked consistent access to food required for an active, healthy life (Coleman-Jensen et al. 2019). Food insecurity rates for households with children were just as troubling. Specifically, 7.1% of U.S. households with children were broadly classified as food insecure with 0.6% of households with children being classified as very low food secure (Coleman-Jensen et al. 2019).

Because of the well-known health and education-related consequences associated with food insecurity, researchers have focused on trying to better understand the core determinants of food insecurity. One area receiving little attention, however, is that of understanding the underlying dynamics associated with food insecurity. Is food insecurity transitory, or is it persistent? Does food insecurity persist across generations? Understanding these dynamics are vital for crafting policy to improve the nutrition and health status of residents in the United States. While important, truly understanding these dynamics is complicated by measurement error in self-reported food security status. Misclassification of food security status in one period can lead to perceived dynamics where none exists or can conceal real transitions in the data.

Confronting measurement error directly, we assess what can be learned about both intra- and intergenerational food security mobility in the United States. Specifically, we apply recent methods developed in Millimet et al. (2020) and Li et al. (2019) on the partial identification of transition matrices when observations may be misclassified. Using multiple waves of data from the University of Michigan's Panel Study of Income Dynamics (PSID) spanning 1999- 2017, the approach bounds transition probabilities under different assumptions concerning misclassification errors and the underlying dynamics. Our main findings are as follows:

- Modest amounts of measurement error leads to bounds on mobility rates that can be quite wide and almost uninformative in the absence of other information or assumptions. Mobility estimates that do not account for measurement error produce a false sense of certitude; observed dynamics in survey data can be highly misleading.

- Assuming that measurement errors are serially uncorrelated over time and misreporting only occurs in the upward direction, and allowing for misclassification of up to 20% of the sample, the estimated *intragenerational* probability of a household being very low food secure (the lowest category of food security) in 2001 (2017) conditional on being very low food secure in 1999 may be as low as 34% (26%). The *intragenerational* probability of a household being food secure (the highest category of food security considered) in 2001 (2017) conditional on being food secure in 1999 is at least 88% (86%).

- Under similar conditions as above, the estimated *intergenerational* probability of an adult child's household being very low food secure in 2017 conditional on the parents' household being very low food secure in 1999 may be as low as 8%; the *intergenerational* probability of an adult child's household being food secure in 2017 conditional on the parents' household being food secure in 1999 is at least 77%.

- Although the estimated transition dynamics are consistent with significant mobility, food security status tends to persist for some households in the tails of the food security distribution (i.e. persistent state of very low food security or persistent state of food security).

- Examining food security dynamics across various subpopulations reveals important heterogeneities. Results provide suggestive evidence of greater mobility, upward and downward, for lower educated households and for non-white households; this holds for both the intra- and intergenerational analyses. Additionally, results provide no evidence of greater upward mobility in high SNAP participation states (though results are not to be interpreted causally).

Researchers need to take seriously the implications of measurement error when estimating food security dynamics. If researchers are willing to invoke assumptions related to the direction and temporal nature of the measurement error process, the estimated bounds on transition probabilities can be narrowed in a transparent way allowing for a clearer understanding of how households, both intra- and intergenerationally, move through the distribution of food security over time.

1 Introduction

Food insecurity is one of the most significant public health concerns facing the United States (Gundersen et al. 2011). Even prior to the COVID-19 pandemic, data from 2018 suggest that 11.1% of U.S. households (14.3 million households) were food insecure; these households lacked consistent access to food required for an active, healthy life (Coleman-Jensen et al. 2019). The figures for households with children are equally concerning. Specifically, 2.7 million households with children (7.1%) were food insecure and another 220,000 households with children (0.6%) were very low food secure (Coleman-Jensen et al. 2019).

The consequences of food insecurity are well documented and affect both children and adults alike. Among children, a lack of requisite nutrition is associated with an array of health and educational-related issues including anemia, increased levels of aggression, and decreased cognitive development (Gundersen 2013). With respect to adults, consequences include attenuated nutrient absorption, deficient dietary needs during pregnancy elevating the risk of birth defects, and general physical and mental health problems (Gundersen 2013). Food insecure elderly adults further face complications related to basic, daily activities (Ziliak et al. 2008).

Because of these complications, researchers have focused on trying to better understand the core determinants of food insecurity; widely accepted factors related to food insecurity include socioeconomic and demographic measures. Specifically, households classified as food insecure are generally those with incomes at or below the federal poverty line and headed by single parents, African-American and Hispanic individuals, and less-educated individuals. Food insecurity is also more prevalent in large cities and rural areas relative to suburban areas and regions on the outskirts of major metropolitan areas.

One area receiving little attention, yet vital for crafting policy to improve the nutrition and health status of residents in the United States, is that of understanding the underlying dynamics associated with food insecurity. Is food insecurity transitory, or is it persistent? Does food insecurity persist across generations, being “passed down” from parents to children? The adverse impacts mentioned above suggest that food insecurity may be persistent; impairments to cognitive development and general physical and mental well-being are likely to reinforce food insecurity. If so, then the effects of food insecurity become magnified as individuals experience prolonged time without consistent access to food. However, the dynamics can also go the opposite direction. If food insecurity is predominantly transitory, then the effects may not be as severe. Perhaps more importantly, whether food insecurity is mostly a permanent or transitory characteristic of households is important for crafting and targeting effective policy and thereby improving household welfare in the long run. With transitory food insecurity, targeted direct food and/or monetary transfers (via food banks or the Supplemental Nutrition Assistance Program (SNAP)) are likely to be sufficient. With persistent food insecurity, more comprehensive interventions that perhaps include health care and/or job training may be needed. Finally, assessing the dynamics of household-level food insecurity is also critical to understanding the nature of observed disparities in food security rates along racial and other dimensions.

While understanding food security dynamics is important, it is complicated by measurement error in self-reported food security status. Although it is fairly well known that accounting for measurement error in self-reported food security status – due to the multidimensional nature of food insecurity, differences in references points, and potential stigma – is important, such error becomes even more salient when assessing dynamics (Duffy and Zizza 2016; Maitra and Rao 2018). On the one hand, misclassification in one period can lead to observed transitions where none exist. For instance, a household that is 'truly' food insecure in two time periods, may report being food secure in the second period due to stigma associated with prolonged food insecurity. On the other hand, misclassification can mask transitions in the observed data. For example, a household that becomes food insecure for the first time in the second time period may continue to report being food secure.

Taking measurement error seriously, we assess what can be learned about both intra- and intergenerational food security mobility in the United States. To do so, we apply recent methods developed in Millimet et al. (2020) and Li et al. (2019) on the partial identification of transition matrices when observations may be misclassified. Using multiple waves of data from the University of Michigan's Panel Study of Income Dynamics (PSID) spanning 1999 – 2017, the approach *bounds* transition probabilities under different assumptions concerning misclassification errors and the underlying dynamics. First, we derive sharp bounds on transition probabilities under minimal assumptions concerning the measurement error process. Second, we narrow the bounds by considering the implications of restrictions on the direction and/or temporal properties of the errors. Finally, we consider monotone instrumental variable (MIV) restrictions that assume monotonic relationships between true food security status and household income (Manski and Pepper 2000).[1]

Our analysis yields some striking findings. First, we show that modest amounts of measurement error leads to bounds on mobility rates that can be quite wide and almost uninformative in the absence of other information or assumptions. This is an important result; mobility estimates that do not account for measurement error produce a false sense of certitude; observed dynamics in survey data can be highly misleading. Second, the assumptions considered here contain significant identifying power as the bounds can be severely narrowed.

Third, under our strongest set of assumptions, but still allowing for misclassification of up to 20% of the sample, we find that the *intragenerational* probability of a household being very low food secure (the lowest category of food security) in 2001 (2017) conditional on being very low food secure in 1999 may be as low as 34% (26%). Conversely, the *intragenerational* probability of a household being food secure (the highest category of food security we consider) in 2001 (2017) conditional on being food secure in 1999 is at least 88% (86%). Under the same set of assumptions, we also find that the *intergenerational* probability of an adult child's household being very low food secure in 2017 conditional on the parents' household being very low food secure in 1999 may be as low as 8%; the *intergenerational* probability of an adult child's household being food secure in 2017 conditional on the parents' household being food secure in 1999 is at least 77%. Thus, while the evidence – both intragenerational and intergenerational – is consistent with significant mobility, we also find food security status to be persistent for at least some households in the tails of the distribution.

Finally, when examining food security dynamics across various subpopulations, we find some evidence of important heterogeneity. We find suggestive evidence of greater mobility – both upward and downward and both intra- and intergenerationally – for lower educated households (defined as high school education and below) and for non-white households. On the other hand, we find little evidence of heterogeneity across states differentiated based on

[1] All coding is performed in Stata and available at `http://faculty.smu.edu/millimet/code.html`.

the state SNAP participation rates among the eligible population. Specifically, we find no evidence of greater upward mobility in high SNAP participation states (although the results are not to be interpreted causally).

Our analysis builds on several prior studies. Gundersen et al. (2019) is most closely related. The authors explore intergenerational food security mobility while confronting endogenous selection into food insecurity during childhood using partial identification methods. Specifically, the authors construct two binary measures of food insecurity for individuals, the first early in life as a child in 1999 and the second during early adulthood in 2015. The authors then partially identify the average treatment effect (ATE) of exposure to food insecurity as a child on adult food security status. Imposing strong but reasonable assumptions, the authors find that growing up in a food insecure household increases the probability of being food insecure as an adult. Our analysis builds on this work in two respects. First, we allow for measurement error in self-reported food security status. Second, we assess finer movements over time in the distribution of food security by partially identifying a 3x3 transition matrix – differentiating between very low food secure, low food secure, and food secure – to provide additional insights into the food security mobility process. However, unlike Gundersen et al. (2019) we do not address endogenous selection into food security states. Thus, our analysis sheds light on persistence in food security within and across generations, but not directly on the causal effect of past food security status on current status.

Two other related studies include McDonough et al. (2019) and Wimer et al. (2019). McDonough et al. (2019) assess intragenerational mobility in food security and compare patterns across white and minority households. The authors find the mobility patterns of Hispanic and non-Hispanic white households with children to be comparable over the long run, whereas non-Hispanic black households with children are more (less) likely to remain food insecure (secure). Wimer et al. (2019) examine intergenerational persistence in food security using the PSID. The authors find that childhood food insecurity is associated with a ten percentage point increase in the likelihood of experiencing food insecurity as an adult even after conditioning on income and wealth. However, McDonough et al. (2019) ignore measurement error altogether, while Wimer et al. (2019) rely on multiple proxies for food security to explore sensitivity related to imprecise measurement. Lastly, our study relates to a number of previous analyses focusing on the relationship between persistence in food insecurity and health- and/or education-related outcomes. These studies differ from ours in that the measurement of mobility is not their objective (Ribar and Hamrick 2003; Jyoti et al. 2005; Hernandez and Jacknowitz 2009; Wilde et al. 2010; Howard 2011).

The rest of the paper unfolds as follows. In Section 2, we presents our empirical approach. Section 3 describes the data. Section 4 presents the results. The final section concludes.

2 Research Methods

2.1 Setup

We follow closely the empirical framework in Millimet et al. (2020) and Li et al. (2019). Thus, we provide only a brief overview and relegate the formal derivations to Appendix A.

To begin, let y^*_{it}, denote the true food security status for household i, $i = 1, ..., N$, in period t, $t = 0, 1$. Define the true $K \times K$ transition matrix as $P^*_{0,1}$, given by

$$P^*_{0,1} = \begin{bmatrix} p^*_{11} & \cdots & \cdots & p^*_{1K} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p^*_{K1} & \cdots & \cdots & p^*_{KK} \end{bmatrix}. \quad (1)$$

Elements of this matrix have the following form

$$\begin{aligned} p^*_{kl} &= \frac{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k,\ \zeta^1_{l-1} \leq y^*_1 < \zeta^1_l)}{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k)} \\ &= \frac{\Pr(y^*_0 \in k,\ y^*_1 \in l)}{\Pr(y^*_0 \in k)} \qquad k, l = 1, ..., K, \end{aligned} \quad (2)$$

where the ζs are cutoff points between the K partitions such that $0 = \zeta^t_0 < \zeta^t_1 < \zeta^t_2 < \cdots < \zeta^t_{K-1} < \zeta^t_K < \infty$, $t = 0, 1$. Thus, p^*_{kl} is a conditional probability. A complete lack of mobility implies p^*_{kl} equals unity if $k = l$ and zero otherwise. We can further define a *conditional* transition matrix, conditioned upon $X = x$, where X denotes a vector of observed attributes. The conditional transition matrix, $P^*_{0,1}(x)$, can be expressed as

$$\begin{aligned} p^*_{kl}(x) &= \frac{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k,\ \zeta^1_{l-1} \leq y^*_1 < \zeta^1_l | X = x)}{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k | X = x)} \\ &= \frac{\Pr(y^*_0 \in k,\ y^*_1 \in l | X = x)}{\Pr(y^*_0 \in k | X = x)} \qquad k, l = 1, ..., K. \end{aligned} \tag{3}$$

Implicit in this definition is the assumption that X includes only time invariant attributes. Further, while the probabilities are conditional on X, the cutoff points ζ are not. Thus, we are capturing movements within the *overall* distribution among those with $X = x$.

In the current setting, we set $K = 3$. The outcome, y^*, denotes household food security status as defined by the USDA. The first partition includes households classified as *very low food secure*. The second partition includes households classified as *low food secure*. Finally, the third partition includes households classified as *food secure*. Note, mobility within the distribution of food security is not zero sum; a household may move up (down) the distribution of food security without another household having to move down (up) the food security distribution. We discuss the definition of y^* in more detail in Section 3.

Our objective is to learn about the elements of $P^*_{0,1}$ or $P^*_{0,1}(x)$. With a random sample $\{y^*_{it}, x_i\}$, the transition probabilities are nonparametrically identified with consistent estimates given by the empirical transition probabilities. However, and as previously noted, there is reasonable concern that food security status is measured with error. As such, let y_{it} denote the *observed* food security status for household i in period t. With the observed data $\{y_{it}, x_i\}$, the empirical transition probabilities are inconsistent for p^*_{kl} and $p^*_{kl}(x)$ given the measurement error. Our objective, then, is to bound the probabilities given in (2) and (3).

To proceed, we characterize the relationships between the true partitions of $\{y^*_{it}\}^1_{t=0}$ and the observed partitions of $\{y_{it}\}^1_{t=0}$ using the following joint probabilities:

$$\theta^{k'l'}_{kl} = \Pr(y_0 \in k',\ y_1 \in l',\ y^*_0 \in k,\ y^*_1 \in l). \tag{4}$$

As noted in Kreider et al. (2012), although conditional misclassification probabilities are perhaps more intuitive, joint probabilities are easier to work with.

In (4) the subscript kl indexes the true status in periods 0 and 1 and the superscript $k'l'$

indicates the observed partitions in the two time periods. θ_{kl}^{kl} represents the probability of no misclassification for a household with true status kl. With this notation, we can rewrite the elements of $P_{0,1}^*$ as

$$
\begin{aligned}
p_{kl}^* &= \frac{\Pr(y_0^* \in k, y_1^* \in l)}{\Pr(y_0^* \in k)} \\
&= \frac{\Pr(y_0 \in k, y_1 \in l) + \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ k'l' \neq kl}} \theta_{kl}^{k'l'} - \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ k'l' \neq kl}} \theta_{k'l'}^{kl}}{\Pr(y_0 \in k) + \sum\limits_{\substack{k',l',\widetilde{l}=1,2,\ldots,K \\ k' \neq k}} \theta_{k\widetilde{l}}^{k'l'} - \sum\limits_{\substack{k',l',\widetilde{l}=1,2,\ldots,K \\ k' \neq k}} \theta_{k'\widetilde{l}}^{kl'}} \\
&\equiv \frac{r_{kl} + Q_{1,kl} - Q_{2,kl}}{p_k + Q_{3,k} - Q_{4,k}} \qquad (5)
\end{aligned}
$$

The expression in (5) is identical to that in Gundersen and Kreider (2008, p. 368). $Q_{1,kl}$ measures the proportion of false *negatives* associated with partition kl (i.e., the joint probability of being misclassified and kl being the true partition). $Q_{2,kl}$ measures the proportion of false *positives* associated with partition kl (i.e., the joint probability of being misclassified and kl being the observed partition). Similarly, $Q_{3,k}$ and $Q_{4,k}$ measure the proportion of false *negatives* and *positives* associated with disability status k, respectively.

The conditional probabilities in (5) are not identified from the data alone. The data identify r_{kl} and p_k (and, hence, $p_{kl} \equiv r_{kl}/p_k$), but not the misclassification parameters, θ. One can compute sharp bounds by searching across the unknown misclassification parameters. There are 72 misclassification parameters in P^*. However, several constraints must hold (Millimet et al. 2020).[2] Even with these constraints, obtaining informative bounds on the transition probabilities is not possible without further restrictions. In Section 2.2, we introduce assumptions on the θs to potentially yield informative bounds. In Section 2.3, we introduce an assumption on the mobility process itself.

[2] Specifically, $Q_{1,kl}+Q_{2,kl} \in [-r_{kl}, 1-r_{kl}]$ and $Q_{3,k}+Q_{4,k} \in [-p_l, 1-p_l]$ $\forall k, l$ to ensure that the numerator and denominator in (5) lie in the unit interval. In addition, $Q_{2,kl} \in [0, r_{kl}]$ $\forall k, l$ as the proportion of false positives associated with partition kl cannot exceed the proportion observed in partition kl. Similarly, $Q_{4,k} \in [0, p_k]$ $\forall k$ as the proportion of false positives associated with partition k cannot exceed the proportion observed in partition k. Finally, $\sum\limits_{\substack{k',\widetilde{k},l'=0,1 \\ l' \neq l}} \theta_{\widetilde{k}l'}^{k'l} \in [0, p_l]$ $\forall l$ as the proportion of false positives associated with partition l cannot exceed the proportion observed in partition l.

2.2 Misclassification

2.2.1 Assumptions

In the presence of measurement error, we obtain bounds on the elements of $P^*_{0,1}$, given in (5).[3] To begin, we consider the following misclassification assumptions from Millimet et al. (2020).

Assumption 1 (Classification-Preserving Measurement Error)**.** *Misreporting does not affect an observation's classified food security status in either the initial or terminal time periods. Formally,* $\sum_{k,l} \theta^{kl}_{kl} = 1$ *or equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\ldots,K \\ k'l' \neq kl}} \theta^{k'l'}_{kl} = 0$$

Assumption 2 (Maximum Arbitrary Misclassification Rate)**.** *The total misclassification rate in the data is bounded from above by* $Q \in (0,1)$*. Formally,* $1 - \sum_{k,l} \theta^{00}_{kl} \leq Q$ *or, equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\ldots,K \\ k'l' \neq kl}} \theta^{k'l'}_{kl} \leq Q.$$

Assumption 3 (Uni-Directional Misclassification)**.** *Misclassification occurs strictly in the upward direction. Formally,*

$$\theta^{k'l'}_{kl} = \begin{cases} \geq 0 \ \text{if } k' \geq k \text{ and } l' \geq l \\ = 0 \ \text{otherwise} \end{cases}.$$

Assumption 1 is a strong assumption that we do not find credible. It simply serves as a benchmark. Assumption 2 places restrictions on the total amount of misclassification allowed in the data. Placement of an upper bound on the probability of a data error in robust estimation is suggested in Horowitz and Manski (1995). In our context, recall there are 72 misclassification parameters. Assumption 2 limits the sum of these parameters, but not the number of parameters. In particular, this assumption does not impose that self-reported

[3] Here we are simply focusing attention on the unconditional transition matrix. In Section 2.3 we discuss the conditional transition matrix.

food security rates overstate true rates or that households that are misclassified in one period are more or less likely to be misclassified in other periods; the misclassification is allowed to be completely arbitrary. In choosing a reasonable value for Q, there is little existing evidence on which to rely. Gundersen and Kreider (2008), exploiting the ordered nature of how questions are asked in the USDA Core Food Security Module (discussed in Section 3), find inconsistency in how questions are answered in 6.1% of the sample. Aside from research on food security, a reasonable amount of measurement error has been documented in areas closely related to food security including SNAP participation, Women, Infants, and Children (WIC) participation, and childcare subsidy receipt (Johnson and Herbst 2013; Kreider et al. 2012). In our baseline analysis, we set $Q = 0.20$, but explore sensitivity to this choice as well.

Assumption 3 restricts the direction of misclassification; household's food security status is classified in either the correct state or a better state. In light of the connection between stigma and food *insecurity*, in particular as it relates to receiving food aid, it is reasonable to think that food insecure households may tend to overstate their level of food security as opposed to food secure households understating their food security status (Duffy and Zizza 2016; Purdam et al. 2016; Witt and Hardin-Fanning 2021). On the other hand, Tadesse et al. (2020) find that households tend to overstate food insecurity when asked about food availability and understate food insecurity when asked about food access. Gundersen and Kreider (2008) find that food secure SNAP recipients may misreport being food insecure due to concerns of losing access to benefits. Invoking this assumption reduces the number of misclassification parameters to 27.

Finally, we consider an additional assumptions from Li et al. (2019). This assumption is imposed in combination with Assumption 2 alone or Assumptions 2 and 3.

Assumption 4 (Temporal Independence)**.** *Misclassification probabilities are independent across time periods. Formally,* $\theta_{kl}^{k'l'}$ *simplifies to*

$$\alpha_k^{k'} \cdot \beta_l^{l'},$$

where $\alpha_k^{k'}$ ($\beta_l^{l'}$) *is the probability of being observed in partition* k' (l') *in the initial (terminal)* *period when the true partition is k (l).*

Assumption 4 restricts misclassification such that the probability of misclassification is independent across the initial and terminal time periods. This effectively rules out a household's historical food security status from affecting the household's propensity to misreport its current food security status. While the measurement error in food security status is potentially serially correlated, for example if stigma persists over time or is passed down from parent to child, the average gap in time between the initial and terminal periods is 10 years (min: 2 years, max: 18 years) in the case of the intragenerational analysis and 17 years (min: 16 years, max: 18 years) in the case of the intergenerational analysis. Given the average interval length for how the data is collected, we believe the assumption is worth exploring in our particular application. Moreover, this assumption is similar in spirit to the assumption of orthogonal errors considered in Gundersen and Kreider (2008). However, in that study, misclassification errors are assumed to be orthogonal to the true value. Here, Assumption 4 restricts the errors to be orthogonal over time. This assumption reduces the number of misclassification parameters to 12. Invoking Assumptions 3 and 4 simultaneously reduces the number of misclassification parameters to six.

2.2.2 Bounds

Under Assumption 1 consistent estimates are given by the empirical transition probabilities (Proposition 1 in Millimet et al. (2020)):

$$\widehat{p}_{kl} = \frac{\sum_i \mathrm{I}(y_{0i} \in k, y_{1i} \in l)}{\sum_i \mathrm{I}(y_{0i} \in k)}.$$

Absent this assumption, the transition probabilities are no longer nonparametrically identified. Bounds under combinations of Assumptions 2 – 4 are detailed in Appendix A.

2.3 Mobility

2.3.1 Assumptions

The preceding section provides bounds on the transition probabilities considering only restrictions on the misclassification process. Here, we introduce restrictions on the mobility process that may further serve to tighten the bounds. The restrictions may be imposed alone or in combination.

Specifically, we consider a monotonicity restriction that places inequality constraints on population transition probabilities across observations with different observed attributes (Manski and Pepper 2000; Chetverikov et al. 2018).

Assumption 5 (Monotonicity). *The conditional probability of upward mobility is weakly increasing in a vector of attributes, u, and the conditional probability of downward mobility is weakly decreasing in the same vector of attributes. Formally, if $u_2 \geq u_1$, then*

$$\begin{aligned} p^*_{kl}(u_1) &\leq p^*_{kl}(u_2) \quad \forall l > k \\ p^*_{kl}(u_1) &\geq p^*_{kl}(u_2) \quad \forall l < k. \\ p^*_{11}(u_1) &\geq p^*_{11}(u_2) \\ p^*_{KK}(u_1) &\leq p^*_{KK}(u_2). \end{aligned}$$

Attributes included in u are referred as monotone instrumental variables (MIV). In our analysis, we let u denote total household income in the initial period for the intragenerational analysis and total parental household income for the intergenerational analysis, where income is discretized into bins corresponding to deciles. We then assume that the probability of upward (downward) mobility through the food security distribution to be no lower (higher) for households with higher incomes and thus higher resources to satisfy requisite food-related needs. Note, the monotonicity assumption provides no information on the conditional staying probabilities, $p^*_{kk}(u)$, for $k = 2, ..., K - 1$.

Implementing the monotonicity restriction requires us to first bound the transition probabilities conditional on u. From (3) and (5), we have

$$p^*_{kl}(u) = \frac{\Pr(y_0 \in k, y_1 \in l|U = u) + \sum_{\substack{k',l'=1,2,...,K \\ k'l' \neq kl}} \theta^{k'l'}_{kl}(u) - \sum_{\substack{k',l'=1,2,...,K \\ k'l' \neq kl}} \theta^{kl}_{k'l'}(u)}{\Pr(y_0 \in k|X = u) + \sum_{\substack{k',l',\tilde{l}=1,2,...,K \\ k' \neq k}} \theta^{k'l'}_{k\tilde{l}}(u) - \sum_{\substack{k',l',\tilde{l}=1,2,...,K \\ k' \neq k}} \theta^{kl'}_{k'\tilde{l}}(u)}$$

$$\equiv \frac{r_{kl}(u) + Q_{1,kl}(u) - Q_{2,kl}(u)}{p_k(u) + Q_{3,k}(u) - Q_{4,k}(u)} \tag{6}$$

where now $Q_{j,\cdot}(u)$, $j = 1, ..., 4$, represent the proportions of false positives and negatives conditional on u. As such, we also consider the following assumption regarding the conditional misclassification probabilities.

Assumption 6 (Independence). *Misclassification rates are independent of the observed attributes of observations, u. Formally,*

$$\theta^{k'l'}_{kl}(u) = \theta^{k'l'}_{kl}, \quad \forall k, k', l, l', u.$$

Assumption 5 may be imposed with or without Assumption 6. However, not imposing Assumption 6 severely limits the identifying power of Assumption 5. This is because

$$\sum_{\substack{k,k',l,l'=0,1 \\ k'l' \neq kl}} \theta^{k'l'}_{kl}(u) \leq Q \qquad \forall u$$

under Assumption 6. However,

$$\sum_{\substack{k,k',l,l'=0,1 \\ k'l' \neq kl}} \theta^{k'l'}_{kl}(u) \leq \frac{Q}{\Pr(U = u)} \qquad \forall u$$

in the absence of Assumption 6.[4]

The plausibility of Assumption 6 depends on one's conjectures concerning the misclassi-

[4]For example, suppose that $Q = 0.1$ and $\Pr(U = u) = 0.2$. Thus, a maximum of 10% of the sample is allowed to be misclassified. Since observations with $U = u$ constitute 20% of the sample, up to 50% (0.1/0.2 = 0.5) of these observations may be misclassified if all observations with $U \neq u$ are correctly classified. However, under Assmption 6, a maximum of 10% of observations may be misclassified with $U = u$ since the misclassification rate is constant for all u.

fication process. However, two points are important to bare in mind. First, the misclassification probabilities, $\theta_{kl}^{k'l'}$, are specific to a pair of true and observed partitions. As a result, even if misclassification is more likely for food insecure households and u is correlated with food security, this does not necessarily invalidate Assumption 6. Second, Assumption 6 does not imply that misclassification rates are independent of all individual attributes, only those included in the variables used to define the level set restrictions.

2.3.2 Bounds

The bounds under various combinations of Assumptions 2 – 5 are relegated to Appendix A.[5] It is important to recognize that the assumptions are mutually exclusive; no assumption implies or is implied by any other assumption and no assumption is incompatible with any other assumption. Assumptions 2, 3, 4, and 6 place restrictions on the misclassification probabilities, θ_{kl}^{kl}. Assumption 2 places an upper limit on their sum. Assumption 4 states that the probability of being misclassified is independent across time periods. Assumption 6 imposes equality of the misclassification probabilities across values of the MIV, total family income. In contrast, Assumption 5 places restrictions on the true conditional probabilities, p_{kl}^{*}. In particular, it restricts the true conditional probabilities to be weakly increasing in total family income.

Regardless of which assumptions are imposed, the estimated bounds suffer from finite sample bias as they rely on infima and suprema (Kreider and Pepper 2007; Millimet et al. 2020). To circumvent this issue, we follow this previous work and utilize a bootstrap bias correction, based on subsampling with replicate samples of size $N/2$. To obtain confidence intervals, we utilize subsampling along with the Imbens-Manski (2004) correction to obtain 90% confidence intervals (CIs).[6] As with the bias correction, we set the size of the replicate samples to $N/2$.

[5] Note, the appendix contains derivations for additional assumptions not considered here.

[6] The literature on inference in partially identified models is expanding rapidly. However, the Imbens-Manski (2004) approach combined with subsampling is preferable in the current context as discussed in Millimet et al. (2020). The preference for subsampling without replacement is due to the fact that the parameters being bounded are probabilities and thus the true value may lie close to the edge of the parameter space.

3 Data

The data comes from the Panel Study of Income Dynamics (PSID), which is the longest running nationally representative longitudinal household survey in the world. The original study dates back to 1968 yielding a sample of over 18,000 individuals comprising approximately 5,000 families residing in the United States. Since the time of the initial survey in 1968, the heads of these original families and heads of any other families subsequently formed by members or posterity of the original 1968 family units have also been followed and surveyed. Given the structure of the survey, the PSID allows for the assessment of food security dynamics over time within a family (intragenerational) as well as over time between parents and adult children (intergenerational).

The data collected on households contains a wealth of information, including information on basic demographics, household composition, health and well-being, educational backgrounds, food security, among others. Of primary interest for this analysis are questions pertaining to food security. In select years the PSID includes the United States Department of Agriculture's (USDA) 18-question Food Security Module (FSM). The first ten questions of the FSM pertain to all households; the final eight questions pertain only to households with children under the age of 18. All questions relate to behaviors and conditions reflecting difficulty in meeting requisite food needs. Moreover, the questions require either a "Yes" or "No" response or offer multiple options. For instance, one question is "'The food that (I/we) bought just didn't last, and (I/we) didn't have money to get more.' Was that often, sometimes, or never true for (you/your household) in the last 12 months?" Another question is "In the last 12 months, were you every hungry but didn't eat because there wasn't enough money for food?" Based on the answers to the FSM and household composition, the USDA classifies households as (i) very low food secure (VLFS), (ii) low food secure (LFS), (iii) marginal food secure (MFS), and (iv) high food secure (HFS). Together, MFS and HFS households comprise food secure (FS) households. For tractability, we focus on transitions between VLFS, LFS, and FS in our analysis.

There are two primary reasons why households may be misclassified. The most obvious is that households intentionally mischaracterize their food situation. This may arise if households overstate their situation due to stigma associated with food insecurity. In addition, households may be misclassified due to differences in reference points. Specifically, answers such as "Often" or "Sometimes" may have different meanings across households and these meanings may, in part, reflect their prior experiences. A third reason for misclassification is that food security is a vague, multidimensional construct and the FSM may represent a flawed attempt at its measurement. This source of misclassification requires a bit more thought. One may define true food security status, y^*, narrowly, following the definition employed by the USDA. For example, a LFS classification for a household with no children requires problematic responses to at least three of ten questions. Taking this as the "true" definition of LFS, then misclassification can only arise due to households answering the questions incorrectly. However, one might alternatively define LFS differently; say, households that at times lack sufficient food required for an active, healthy life. With such a "true" definition of LFS, misclassification may arise not only due to households answering the questions incorrectly, but also because the questions do not completely map to this definition. One of the advantages of our approach is that either interpretation is acceptable. That said, the interpretation does have implications for thinking about the maximum degree of misclassification in the data, Q. Under the first interpretation, there are only two sources of misclassification and Q may be relatively small. Under the second interpretation, there are three sources of misclassification. Thus, a relatively large value of Q may be a more reasonable choice.

The particular waves of the data used to estimate the transition dynamics include the 1999, 2001, 2003, 2015, and 2017 waves for the intragenerational analysis and the 1999, 2015, and 2017 waves for the intergenerational analysis. In estimating the intergenerational food security dynamics, the 1999 wave corresponds to adult parents with the 2015 and 2017 waves corresponding to adult children. Additionally for the intergenerational analysis, we restrict the sample to households with dependants/children 18 years or younger in 1999.[7]

In constructing the estimation samples, observations are retained for those with valid

[7]Specifically, the sample for the intergenerational analysis includes households with individuals related to the head of household (reference person) coded as either the son or daughter of the reference person, the stepson or stepdaughter of the reference person, or the son or daughter of the partner linked to the reference person.

measures of household-level food security in both the initial and terminal time periods. The samples are further restricted to observations with valid measures of total household income, the monotone instrumental variable (MIV) used in the analysis. For the intergenerational analysis, estimation samples are constructed two ways. The sample is first constructed restricting the adult child to being the head of household in the terminal period. The sample is then expanded allowing adult children to be either the head of household or the spouse/partner of the head of household in the terminal period. Doing such results in multiple balanced panels with estimation sample sizes of 12,328 (6,164 unique individuals; 1999 to 2001 panel), 11,676 (5,838 unique individuals; 1999 to 2003 panel), 7,450 (3,725 unique individuals; 1999 to 2015 panel), and 7,044 (3,522 unique individuals; 1999 to 2017 panel) for the intragenerational analysis. The estimation sample sizes for the intergenerational analysis when adult children are restricted to being the head of household are 3,578 (1,789 adult parent-adult child pairs; 1999 to 2015 panel) and 3,852 (1,926 adult parent-adult child pairs; 1999 to 2017 panel). When not restricting adult children to being the head of household in the terminal period, the estimation sample sizes increase to 4,742 (2,371 adult parent-adult child pairs; 1999 to 2015 panel) and 5,216 (2,608 adult parent-adult child pairs; 1999 to 2017 panel). Lastly, when imposing monotonicity restrictions, we use total household income expressed in deciles.

The average food insecurity rate across all waves of data and analyses is 13.2%. Average household income (nominal) is $61,926. The average age of the head of household is approximately 41 years old with an average family size of three. The average number of years of schooling for the heads of households, across all samples, is approximately 13 years. Lastly, approximately 55% of household heads identify as white while 45% identify as non-white. Detailed descriptive statistics broken down by analysis type and years can be found in Table 1.

4 Results

4.1 Full Sample

Results obtained pooling all households are presented in Tables 2-3 and Figures 1-3. Figures B1-B4 and Figures C1-C4 in the supplemental appendix contain additional results.[8] Table 2 and Figures 1-2 contain the results for intragenerational mobility. Table 3 and Figure 3 contain the results for intergenerational mobility. Each table contains five panels and displays the point estimates for the bounds along with 90% confidence intervals. Panel I contains the estimates of the conditional probabilities under the assumption of no misclassification (Assumption 1). Panel II displays the bounds under Assumption 2, setting $Q = 0.20$. Panel III adds the assumption of uni-directional misclassification (Assumption 3). Lastly, Panels IV and V combine Assumptions 2 and 3 with the assumption of temporal invariance (Assumption 4) and monotonicity (Assumptions 5 and 6), respectively.[9]

For intragenerational mobility, the left side of the Table 2 displays the transition matrix over a relatively short time period, from 1999 to 2001. Figure 1 displays these results graphically, showing how the bounds vary as Q increases from 0 to 0.30. The right side of the Table 2 displays the transition matrix over a longest time period observed, from 1999 to 2017. Figure 2 is analogous to Figure 1 for this extended time period. Figures B1-B4 in the supplemental appendix displays the bounds graphically under different sets of assumptions and over different time periods.

For intergenerational mobility, the left side of the Table 3 displays the transition matrix over the time period from 1999 to 2017, restricting the sample to pairs where the adult child is the head of household. The right side of the Table 3 expands the sample to include any adult child. Figure 3 displays the results graphically for the sample of adult children that are household heads, showing how the bounds vary as Q increases from 0 to 0.30. Figures C1-C4 in the supplemental appendix displays the bounds graphically under different sets of assumptions and over different time periods for both samples of adult children.

[8] All figures display the point estimates of the bounds.

[9] For brevity, we do not report bounds based on all possible combinations of restrictions. Unreported results are available upon request.

4.1.1 Intragenerational Analysis

Panel I in Table 2 presents our baseline results under the strong assumption of Classification-Preserving Measurement Error (Assumption 1). Maintaining this assumption, the data indicate that households are at least somewhat upwardly mobile, especially over longer time periods. Conditional on being VLFS in 1999, the probability of a household being LFS in 2001 (2017) is 23.1% (19.8%); the probability of a household being FS in 2001 (2017) is 37.6% (54.2%).[10] Conditional on being LFS in 1999, the probability of a household being FS in 2001 (2017) is 59.6% (70.8%). That said, the conditional staying probability for VLFS in 2001 (2017) is 39.3% (26.0%). The probability of remaining LFS or becoming VLFS in 2001 (2017) conditional on being LFS in 1999 is 40.3% (29.1%).

While these figures suggest that food insecurity has both a sizeable permanent and transitory component, it is important to see what can be learned when Assumption 1 is relaxed. Panel II in Table 2 presents the so-called *worst case* bounds obtained only under Assumption 2. Here we see that the bounds are completely uninformative for the transition probabilities conditional on being VLFS or LFS in the initial period. This arises because less than 8% (3%) of the sample reports being LFS (VLFS) in 1999. Thus, allowing for an arbitrary 20% of the sample to be misclassified is sufficient to preclude one from learning anything from the data absent additional information. Although discouraging, the importance of this finding cannot be overstated as it indicates that mistakenly believing mobility estimates that do not account for measurement error gives a false sense of certitude. For example, while based on the assumption of no misclassification error (or rank preserving misclassification error), one may be tempted to believe that probability of a household being VLFS in 1999 and 2017 is 26.0%, the true probability is completely unknown with $Q = 0.20$.

The worst case bounds are fairly informative for the transition probabilities conditional on being FS in the initial period. As roughly 90% of the sample reports being FS in 1999, one can learn much even when allowing for an arbitrary 20% of the sample to be misclassified. Specifically, we find that the conditional staying probability for FS from 1999 to 2001 (2017) is at least 73.6% (70.8%). The probability of transitioning for FS in 1999 to LFS in 2001

[10]Throughout the discussion of the results, unless otherwise noted, we focus on the point estimates for simplicity. The confidence intervals are generally not much wider than the point estimates of the bounds.

(2017) may be as high as 25.6% (26.5%). The probability of transitioning for FS in 1999 to VLFS in 2001 (2017) may be as high as 23.0% (25.0%).

To see what can be learned if one is willing to invoke stronger assumptions, we turn to Panel III which adds the assumption of uni-directional errors. Adding this assumption does not affect the bounds on the transition probabilities conditional on being LFS or FS in the initial period. However, it does have some identifying power for the transition probabilities conditional on being VLFS in the initial period. Most importantly, we now find that the conditional staying probability for VFLS in 2001 (2017) is at least 4.8% (3.1%). Thus, under the relatively weak assumptions of 20% misclassification and uni-directional measurement error, we are able to reject the hypothesis that VLFS is completely transitory. More precisely, we can rule out the possibility that all households who are VLFS in 1999 are no longer VLFS in 2001 or 2017.

Panel IV adds the assumption of temporal independence. This assumption has significant identifying power as the bounds tighten considerably. First, the conditional staying probability for VLFS in 2001 (2017) is now at least 34.1% (26.0%). Moreover, the probability of transitioning from VLFS in 1999 to LFS in 2001 (2017) is no more than 65.9% (74.0%); the probability of transitioning from VLFS in 1999 to FS in 2001 (2017) is no more than 37.6% (54.2%). Second, the conditional staying probability for FS in 2001 (2017) is now at least 88.4% (85.6%). Moreover, the probability of transitioning from FS in 1999 to LFS in 2001 (2017) is no more than 10.8% (11.6%); the probability of transitioning from FS in 1999 to VLFS in 2001 (2017) is no more than 8.2% (10.2%). While interesting and informative, it is noteworthy that under this set of assumptions, we still learn nothing about the transition probabilities conditional on being LFS in 1999. Moreover, we cannot rule out the possibility that all three conditional staying probabilities are equal to one.[11]

Panel V replaces the assumption of temporal independence with the MIV assumption. When $Q = 0.20$ this assumption has no identifying power; the bounds are identical to those in Panel III.

[11] It is important to recognize that while we cannot rule out the possibility that each conditional staying probability *individually* equals one, we cannot say whether one can rule out the possibility that all three conditional staying probabilities are *simultaneously* equal to one. To do so requires one to bound multiple parameters simultaneously – a substantially more difficult task – and we do not pursue this here.

Figures 1 and 2 show graphically how the bounds vary with the maximum misclassification rate, Q. The figures also allow for visual inspection of the identifying power of each assumption. A few findings stand out. First, the worst case bounds on the transition probabilities conditional on initially being VLFS or LFS quickly widen as Q increases. In the majority of cases, the bounds become completely uninformative when Q is at least 0.05. Second, the assumptions about uni-directional and temporally independent errors both have identifying power. However, the monotonicity assumption only has modest identifying power, and even then only when Q is small. Third, there are a few instances where the bounds change noticeably when the terminal period changes from 2001 (Figure 1) to 2017 (Figure 2). For example, upper bound on the conditional staying probability of LFS is roughly 60% in Figure 2 when $Q = 0.10$; in comparison, the upper bound is roughly 75% in Figure 1. Conversely, lower bound on the probability of transitioning from LFS to FS is roughly 10% in Figure 1 when $Q = 0.10$; in comparison, the lower bound is roughly 25% in Figure 2. Finally, our analysis makes it clear that it is difficult to learn much about the dynamics of food insecure households in the presence of modest misclassification given the small percentage of households self-identifying in these categories. This is especially the case if classification errors are serially correlated.

4.1.2 Intergenerational Analysis

Panel I in Table 3 presents our baseline results for intergenerational mobility under the strong assumption of Classification-Preserving Measurement Error (Assumption 1). With this assumption, the data show significant upward mobility and only modest association between parental and adult child food security. Conditional on parents being VLFS in 1999, the probability of an adult child being FS in 2017 is 72.4% (73.3%) when the sample is (is not) restricted to adult children who are heads of households; the probability increases to 82.6% (84.8%) conditional on the parents being FS in 1999. Conversely, the probability of an adult child being VLFS in 2017 conditional on parents being VLFS in 1999 is 9.2% (7.8%) when the sample is (is not) restricted to adult children who are heads of households; the probability decreases to 7.2% (6.0%) conditional on the parents being FS in 1999.

While these figures suggest a modest intergenerational association in food security, again we must see what can be learned when Assumption 1 is relaxed. Panel II in Table 3 presents the worst case bounds obtained under Assumption 2 with $Q = 0.20$. As in Table 2, the bounds are completely uninformative for the transition probabilities conditional on parents being VLFS or LFS in 1999. Here, roughly 11% (4%) of the parent households report being LFS (VLFS) in 1999. As with the analysis of intragenerational mobility, this shows the false certainty that one can obtain about the nature of food security dynamics when ignoring measurement error.

The worst case bounds provide some information on the transition probabilities conditional on parents being FS in the initial period. As roughly 85% of the sample of parents reports being FS in 1999, the data are useful even when allowing for an arbitrary 20% of the sample to be misclassified. Specifically, we find that the intergenerational conditional staying probability for FS is at least 59.1% (61.6%) when the sample is (is not) restricted to adult children who are heads of households. The probability of adult children being LFS in 2017 despite parents being FS in 1999 may be as high as 33.8% (32.4%) when the sample is (is not) restricted to adult children who are heads of households. Similarly, the upper bound on the probability of adult children being VLFS in 2017 despite parents being FS in 1999 is 30.7% (29.2%) when the sample is (is not) restricted to adult children who are heads of households.

Panel III adds the assumption of uni-directional errors. Adding this assumption does not affect the bounds on the transition probabilities conditional on parents being LFS or FS in the initial period. However, it does have a minor amount of identifying power for the transition probabilities conditional on parents being VLFS in the initial period. Specifically, under these relatively weak assumptions, the lower bounds on the intergenerational conditional staying probability for VFLS is at least 1.5% (1.1%) when the sample is (is not) restricted to adult children who are heads of households. Thus, there is at least some intergenerational association in VLFS status. This is an important finding; combining the data with only Assumptions 2 and 3 is sufficient to uncover (statistically significant) evidence of intergenerational correlation in food insecurity.

Panel IV adds the assumption of temporal independence. As in Table 2, this assumption has significant identifying power. First, the intergenerational conditional staying probability for VLFS is now at least 7.8% across the two samples. Moreover, the probability of an adult child being LFS (FS) conditional on parents being VLFS is no more than 90.8% (72.4%) across the two samples. Second, bounds on the intergenerational conditional staying probability for FS are $[0.748, 0.979]$ ($[0.770, 0.985]$) when the sample is (is not) restricted to adult children who are heads of households. Moreover, bounds on the probability of an adult child being LFS conditional on parents being for FS are $[0.024, 0.181]$ ($[0.015, 0.169]$) when the sample is (is not) restricted to adult children who are heads of households. Thus, at least approximately 1.5%, but no more than roughly 18.1%, of all adult children are LFS despite their parents being FS. Third, the upper bound on the intergenerational conditional staying probability for LFS is roughly 80% across the two samples; the lower bound on the probability of adult children being FS conditional on parents being LFS is at least 11.6%. This is a noteworthy finding. Under relatively weak assumptions, at least 12-15% of adult children are FS despite their parents experiencing LFS. Finally, we can reject the possibility of no intergenerational mobility as the upper bounds on the intergenerational conditional staying probabilities are less than one for LFS and FS in both samples.

Panel V replaces the assumption of temporal independence with the MIV assumption. When $Q = 0.20$ this assumption has very modest identifying power on transition probabilities for adult children whose parents are VLFS; the bounds are nearly identical to those in Panel III. There is some identifying power on the upper bound of the probability of an adult child being VLFS in 2017 conditional on parents being LFS in 1999; the estimate falls to 85.7% (88.1%) when the sample is (is not) restricted to adult children who are heads of households.

Figure 3 plots the bounds as a function of the maximum misclassification rate, Q, as in Figures 1 and 2. A few findings stand out. First, the worst case bounds on the transition probabilities conditional on parents being VLFS or LFS quickly widen as Q increases. While this was also the case in the analysis of intragenerational mobility, it is not quite as dramatic here as the bounds occasionally do not become completely uninformative until Q reaches 0.10. Second, the assumptions about uni-directional and temporally independent errors both have identifying power. In contrast to the analysis of intragenerational mobility, the monotonicity assumption often has some identifying power, even when Q is of moderate size. Finally, as with intragenerational mobility, our analysis showcases the difficulty in estimating intergenerational dynamics among food insecure households in the presence of modest misclassification given the small percentage of households classified as such in the observed data. This is particularly true if classification errors are correlated across generations.

4.2 Sub-Samples

To assess heterogeneity in food security dynamics, we estimate bounds under the preceding assumptions conditional on X. In practice, this is accomplished by simply splitting the sample. We conduct three such analyses, splitting the sample by race (white and non-white), education (high school or less versus at least some college), and state SNAP participation (states where the average SNAP participation among the eligible population over the period 2007-2009 is below and above the median). For brevity, we only display select results for the conditional staying probabilities in Figures 4-9. In particular, the bounds are assess over the period from 1999 to 2017 for both the intragenerational and intergenerational analysis. Moreover, for the intergenerational analysis, we focus on the sample of adult children who are heads of household. Figures B5-10 and C5-C10 provide additional results.

4.2.1 Race

Figures 4 and 5 display the bounds by race for intragenerational and intergenerational conditional staying probabilities, respectively. A few interesting differences do emerge across racial lines. First, upper bounds on the *intragenerational* conditional staying probabilities for LFS and FS are generally lower for non-whites. For example, whereas the worst case bounds for whites include one when Q is as small as 0.05, they do not include one until Q is 0.10 (0.15) for LFS (FS) for non-whites. Moreover, imposing the assumptions of uni-directional and temporally independent errors and assuming Q is 0.10, the upper bound on the conditional staying probability for LFS is one for whites, but only about 0.42 for non-whites.

Second, under these same assumptions but assuming $Q = 0.20$, bounds on the *intragenerational* conditional staying probability for FS are roughly $[0.88, 1.00]$ for whites, but roughly $[0.80, 0.99]$ for non-whites. Although we can rule out the possibility that the probabilities are equal (or even higher for non-whites) since the bounds overlap, this is not the case at lower values of Q. For instance, if $Q = 0.05$, the bounds are strictly higher for whites; roughly $[0.94, 0.99]$ for whites and $[0.85, 0.91]$ for non-whites.

Third, upper bounds on the *intergenerational* conditional staying probabilities for LFS and FS are also generally lower for non-whites. While the worst case bounds for whites include one when Q is as small as 0.10 (0.15) for LFS (FS), they do not include one until Q is 0.15 (0.20) for non-whites. Moreover, imposing the assumptions of uni-directional and temporally independent errors and assuming Q is 0.20, the upper bound on the conditional staying probability for LFS is one for whites, but only about 0.60 for non-whites.

Finally, under these same assumptions, bounds on the *intergenerational* conditional staying probability for FS are roughly $[0.80, 1.00]$ for whites, but roughly $[0.70, 0.90]$ for non-whites. However, if $Q = 0.10$, the bounds are strictly higher for whites; roughly $[0.84, 0.95]$ for whites and $[0.73, 0.83]$ for non-whites. thus, with even a relatively modest amount of misclassification, we can reject equality of the probability of adult children being FS conditional on their parents being FS.

Overall, the results (along with the more detailed results in the supplemental appendix) suggest greater mobility – both upward and downward and both intra- and intergenerationally – for non-whites than whites.

4.2.2 Education

Figures 6 and 7 display the bounds by education for intragenerational and intergenerational conditional staying probabilities, respectively. Again, a few interesting differences appear across education groups. First, the results for *intragenerational* mobility suggest greater mobility among lower educated households. For example, under the assumptions of uni-directional and temporally independent errors, the upper bound on the conditional staying probability for LFS is roughly 0.50 (0.80) when $Q = 0.10$ ($Q = 0.20$) for lower educated households. For higher educated households, the upper bound is essentially one for all values of Q greater than 0.10.

Second, under these same assumptions and assuming $Q = 0.20$, bounds on the *intragenerational* conditional staying probability for FS are roughly $[0.85, 0.96]$ for lower educated households, but roughly $[0.93, 0.99]$ for higher educated households. At lower values of Q, the bounds no longer overlap. For example, if $Q = 0.05$, the bounds are strictly higher for higher educed households; roughly $[0.87, 0.92]$ for the lower educated and $[0.95, 0.98]$ for the higher educated.

Third, bounds on the *intergenerational* conditional staying probabilities for all levels of food security are generally lower for the less educated. Under the assumptions of uni-directional and temporally independent errors, the upper bound on the conditional staying probability for VLFS is much lower for the less educated. For example, the upper bound is roughly 0.60 for lower educated households, but one for higher educated households, when $Q = 0.10$. A similar pattern holds for LFS. For FS, the bounds barely overlap even when Q is as high as 0.20. At this level of misclassification, bounds on the conditional staying probability for FS are roughly $[0.62, 0.85]$ for lower educated households, but roughly $[0.83, 0.99]$ for higher educated households.

Overall, the results (along with the more detailed results in the supplemental appendix) suggest greater mobility – both upward and downward and both intra- and intergenerationally – for lower educated households. The fact that this is similar to the heterogeneity along racial lines is not surprising given the correlation between race and education.

4.2.3 SNAP

Figures 8 and 9 display the bounds by state SNAP participation for intragenerational and intergenerational conditional staying probabilities, respectively.[12] Interestingly, the estimated bounds are nearly identical in both the intra- and intergenerational analyses. The sole exception is when examining the *intergenerational* conditional staying probabilities for VLFS. First, the observed conditional staying probability is much higher in high SNAP participation states; roughly 22% as opposed to only about 5% in low SNAP participation states. Second, the upper bound under the assumptions of uni-directional and temporally independent errors is often lower in low SNAP participation states. For example, the upper bound on the conditional staying probability is roughly 0.40 (0.70) when $Q = 0.05$ ($Q = 0.10$) for in low SNAP participation states, it is roughly 0.83 (1.00) in high SNAP participation states. While these results should not be interpreted causally, it is noteworthy that we fail to find

[12] State-level SNAP participation rates for all eligible individuals comes from Cunnyngham et al. (2012).

evidence of greater upward mobility in high SNAP participation states.

5 Discussion & Conclusion

Food insecurity is a considerable public health concern faced by millions of individuals in the United States today. In light of the well known health concerns associated with irregular access to food required for an active, healthy life, researchers have focused attention on understanding the key determinants of food insecurity. In addition to understanding the core determinants, understanding the underlying dynamics are equally important for crafting policy aimed at improving the health and nutritional status of food insecure individuals. While understanding these dynamics are important, doing so is complicated by measurement error in self-reported food security status.

Using data from the University of Michigan's Panel Study of Income Dynamics (PSID) spanning 1999- 2017, we confront measurement error directly and assess what can be learned about both intra- and intergenerational food security mobility in the United States. Specifically, we apply recent methods developed in Millimet et al. (2020) and Li et al. (2019) and partially identify transition matrices under various scenarios of misreported food security status in the data. We find that just a modest amount of measurement error leads to estimated bounds on food security mobility rates that can be quite wide and almost uninformative in the absence of other information or assumptions. If one is willing to assume that a household's self-reported food security status reflects either the true or better state (i.e. misreporting only in the upward direction) and willing to rule out a household's historical food security status from affecting the household's propensity to misreport its current food security status (i.e. serially uncorrelated measurement errors), then the bounds can be drastically tightened. Informative bounds on the transition probabilities can be inferred even in the presence of a nontrivial amount of measurement error.

The transition dynamics estimated and presented here are consistent with significant mobility through the food security distribution over time (both intra- and intergenerationally), although food security status for some households in tails of the food security distribution tends to persist. Exploring the dynamics across various subpopulations provides some evidence of greater mobility, both upward and downward, for lower educated and non-white households; this holds for both the intra- and intergenerational analyses. Lastly, results provide no evidence of greater upward mobility in high SNAP participation states though these results are not intended to be interpreted causally.

Researchers need to take seriously the implications of measurement error when estimating food security dynamics. If researchers are willing to invoke assumptions related to the direction and temporal nature of the measurement error process, the estimated bounds on transition probabilities can be narrowed in a transparent way allowing for a clearer understanding of how households, both intra- and intergenerationally, move through the distribution of food security over time.

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Table 1. Descriptive Statistics

	1999 - 2001		1999 - 2003		1999 - 2015		1999 - 2017	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Intragenerational Food Security Dynamics								
Food Insecure (1 = food insecure)	0.092	0.289	0.086	0.280	0.112	0.315	0.100	0.299
Total Household Income ($1000s, nominal)	58.046	69.795	58.785	76.481	70.563	79.626	72.132	84.260
White (1 = HoH white)	0.616	0.486	0.615	0.487	0.604	0.489	0.606	0.489
Non-White (1 = HoH non-white)	0.384	0.486	0.385	0.487	0.396	0.489	0.394	0.489
Age (years)	45.609	15.540	46.429	15.253	49.438	14.710	50.150	15.081
Education (years)	12.740	2.882	12.764	2.854	13.116	2.806	13.172	2.766
Family Size	2.793	1.516	2.800	1.508	2.693	1.474	2.653	1.459
Number of Observations	12,328		11,676		7,450		7,044	
Intergenerational Food Security Dynamics - HoH								
Food Insecure (1 = food insecure)	-	-	-	-	0.184	0.388	0.169	0.375
Total Household Income ($1000s, nominal)	-	-	-	-	53.856	67.203	57.488	66.797
White (1 = HoH white)	-	-	-	-	0.472	0.499	0.467	0.499
Non-White (1 = HoH non-white)	-	-	-	-	0.528	0.499	0.533	0.499
Age (years)	-	-	-	-	33.775	8.289	34.096	7.887
Education (years)	-	-	-	-	13.172	2.595	13.230	2.603
Family Size	-	-	-	-	3.274	1.811	3.291	1.792
Number of Observations	-		-		3,578		3,852	
Intergenerational Food Security Dynamics - Any								
Food Insecure (1 = food insecure)	-	-	-	-	0.163	0.369	0.151	0.358
Total Household Income ($1000s, nominal)	-	-	-	-	60.066	74.694	64.471	79.490
White (1 = HoH white)	-	-	-	-	0.517	0.500	0.516	0.500
Non-White (1 = HoH non-white)	-	-	-	-	0.483	0.500	0.484	0.500
Age (years)	-	-	-	-	34.171	8.351	34.514	7.934
Education (years)	-	-	-	-	13.207	2.606	13.255	2.619
Family Size	-	-	-	-	3.433	1.757	3.476	1.751
Number of Observations	-		-		4,742		5,216	

Notes: SD = standard deviation; HoH = adult-child is head of household; Any = adult-child is head of household *or* spouse/partner of head of household

Table 2. Bounds on Intragenerational Food Security Transition Probabilities.

	1999 -- 2001				1999 -- 2017		
I. No Misclassification							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.393,0.393]	[0.231,0.231]	[0.376,0.376]	**VLFS**	[0.260,0.260]	[0.198,0.198]	[0.542,0.542]
	(0.332,0.457)	(0.171,0.285)	(0.323,0.430)		(0.173,0.340)	(0.124,0.269)	(0.460,0.626)
LFS	[0.104,0.104]	[0.299,0.299]	[0.596,0.596]	**LFS**	[0.121,0.121]	[0.170,0.170]	[0.708,0.708]
	(0.081,0.127)	(0.263,0.331)	(0.561,0.630)		(0.091,0.153)	(0.135,0.210)	(0.657,0.749)
FS	[0.008,0.008]	[0.034,0.034]	[0.958,0.958]	**FS**	[0.027,0.027]	[0.042,0.042]	[0.931,0.931]
	(0.006,0.010)	(0.030,0.038)	(0.953,0.962)		(0.023,0.032)	(0.036,0.048)	(0.924,0.938)
II. Misclassification (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**VLFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.230]	[0.000,0.256]	[0.736,1.000]	**FS**	[0.000,0.250]	[0.000,0.265]	[0.708,1.000]
	(0.000,0.233)	(0.000,0.261)	(0.731,1.000)		(0.000,0.255)	(0.000,0.270)	(0.701,1.000)
III. Misclassification + Uni-Directional Errors (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.048,1.000]	[0.000,0.952]	[0.000,0.923]	**VLFS**	[0.031,1.000]	[0.000,0.969]	[0.000,0.945]
	(0.039,1.000)	(0.000,0.961)	(0.000,0.935)		(0.019,1.000)	(0.000,0.981)	(0.000,0.959)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.230]	[0.000,0.256]	[0.736,1.000]	**FS**	[0.000,0.250]	[0.000,0.265]	[0.708,1.000]
	(0.000,0.233)	(0.000,0.261)	(0.731,1.000)		(0.000,0.255)	(0.000,0.270)	(0.701,1.000)
V. Misclassification + Uni-Directional Errors + Temporal Independence (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.341,1.000]	[0.000,0.659]	[0.000,0.376]	**VLFS**	[0.260,1.000]	[0.000,0.740]	[0.000,0.542]
	(0.332,1.000)	(0.000,0.668)	(0.000,0.430)		(0.173,1.000)	(0.000,0.827)	(0.000,0.626)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.082]	[0.000,0.108]	[0.884,1.000]	**FS**	[0.000,0.102]	[0.000,0.116]	[0.856,1.000]
	(0.000,0.084)	(0.000,0.112)	(0.879,1.000)		(0.000,0.106)	(0.000,0.122)	(0.850,1.000)
IV. Misclassification + Uni-Directional Errors + Monotonicity (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.048,1.000]	[0.000,0.952]	[0.000,0.923]	**VLFS**	[0.031,1.000]	[0.000,0.969]	[0.000,0.945]
	(0.039,1.000)	(0.000,0.961)	(0.000,0.935)		(0.022,1.000)	(0.000,0.978)	(0.000,0.959)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.230]	[0.000,0.256]	[0.736,1.000]	**FS**	[0.000,0.250]	[0.000,0.265]	[0.708,1.000]
	(0.000,0.233)	(0.000,0.261)	(0.731,1.000)		(0.000,0.255)	(0.000,0.270)	(0.701,1.000)

Notes: VLFS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Point estimates for bounds provided in brackets obtained using 50 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 200 subsamples of size N/2. Monotonicity restrictions based on family non-labor income. See text for further details.

Table 3. Bounds on Intergenerational Food Security Transition Probabilities.

	Household Head				Any Child		
I. No Misclassification							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.092,0.092]	[0.184,0.184]	[0.724,0.724]	**VLFS**	[0.078,0.078]	[0.189,0.189]	[0.733,0.733]
	(0.045,0.143)	(0.114,0.263)	(0.636,0.816)		(0.024,0.125)	(0.128,0.261)	(0.653,0.809)
LFS	[0.083,0.083]	[0.176,0.176]	[0.741,0.741]	**LFS**	[0.082,0.082]	[0.153,0.153]	[0.765,0.765]
	(0.049,0.112)	(0.131,0.222)	(0.695,0.786)		(0.056,0.109)	(0.116,0.189)	(0.727,0.808)
FS	[0.072,0.072]	[0.102,0.102]	[0.826,0.826]	**FS**	[0.060,0.060]	[0.092,0.092]	[0.848,0.848]
	(0.060,0.081)	(0.090,0.116)	(0.811,0.843)		(0.053,0.068)	(0.083,0.103)	(0.835,0.858)
II. Misclassification (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**VLFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.307]	[0.000,0.338]	[0.591,1.000]	**FS**	[0.000,0.292]	[0.000,0.324]	[0.616,1.000]
	(0.000,0.318)	(0.000,0.353)	(0.575,1.000)		(0.000,0.300)	(0.000,0.337)	(0.603,1.000)
III. Misclassification + Uni-Directional Errors (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.015,1.000]	[0.000,0.985]	[0.000,0.955]	**VLFS**	[0.011,1.000]	[0.000,0.989]	[0.000,0.961]
	(0.008,1.000)	(0.000,0.992)	(0.000,0.970)		(0.003,1.000)	(0.000,0.997)	(0.000,0.974)
LFS	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,1.000]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.307]	[0.000,0.338]	[0.591,1.000]	**FS**	[0.000,0.292]	[0.000,0.324]	[0.616,1.000]
	(0.000,0.318)	(0.000,0.353)	(0.575,1.000)		(0.000,0.300)	(0.000,0.337)	(0.603,1.000)
IV. Misclassification + Uni-Directional Errors + Temporal Independence (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.092,1.000]	[0.000,0.908]	[0.000,0.724]	**VLFS**	[0.078,1.000]	[0.000,0.922]	[0.000,0.733]
	(0.045,1.000)	(0.000,0.955)	(0.000,0.816)		(0.024,1.000)	(0.000,0.976)	(0.000,0.809)
LFS	[0.000,0.678]	[0.000,0.771]	[0.146,1.000]	**LFS**	[0.000,0.731]	[0.000,0.802]	[0.116,1.000]
	(0.000,0.748)	(0.000,0.855)	(0.062,1.000)		(0.000,0.812)	(0.000,0.876)	(0.040,1.000)
FS	[0.000,0.150]	[0.024,0.181]	[0.748,0.976]	**FS**	[0.000,0.138]	[0.015,0.169]	[0.770,0.985]
	(0.000,0.161)	(0.011,0.194)	(0.732,0.989)		(0.000,0.145)	(0.006,0.181)	(0.758,0.994)
V. Misclassification + Uni-Directional Errors + Monotonicity (Q = 0.20)							
	VFLS	**LFS**	**FS**		**VFLS**	**LFS**	**FS**
VLFS	[0.019,1.000]	[0.000,0.981]	[0.000,0.955]	**VLFS**	[0.017,1.000]	[0.000,0.983]	[0.000,0.952]
	(0.008,1.000)	(0.000,0.992)	(0.000,0.962)		(0.005,1.000)	(0.000,0.995)	(0.000,0.965)
LFS	[0.000,0.857]	[0.000,1.000]	[0.000,1.000]	**LFS**	[0.000,0.881]	[0.000,1.000]	[0.000,1.000]
	(0.000,1.000)	(0.000,1.000)	(0.000,1.000)		(0.000,1.000)	(0.000,1.000)	(0.000,1.000)
FS	[0.000,0.307]	[0.000,0.338]	[0.591,1.000]	**FS**	[0.000,0.292]	[0.000,0.323]	[0.616,1.000]
	(0.000,0.318)	(0.000,0.344)	(0.575,1.000)		(0.000,0.300)	(0.000,0.327)	(0.604,1.000)

Notes: VLFS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. First (second) generation measured in 1999 (2017). Household Head (Any Child) refers to the fact that the second generation, adult child is (not) restricted to be the head of their household. Point estimates for bounds provided in brackets obtained using 50 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 200 subsamples of size N/2. Monotonicity restrictions based on family non-labor income. See text for further details.

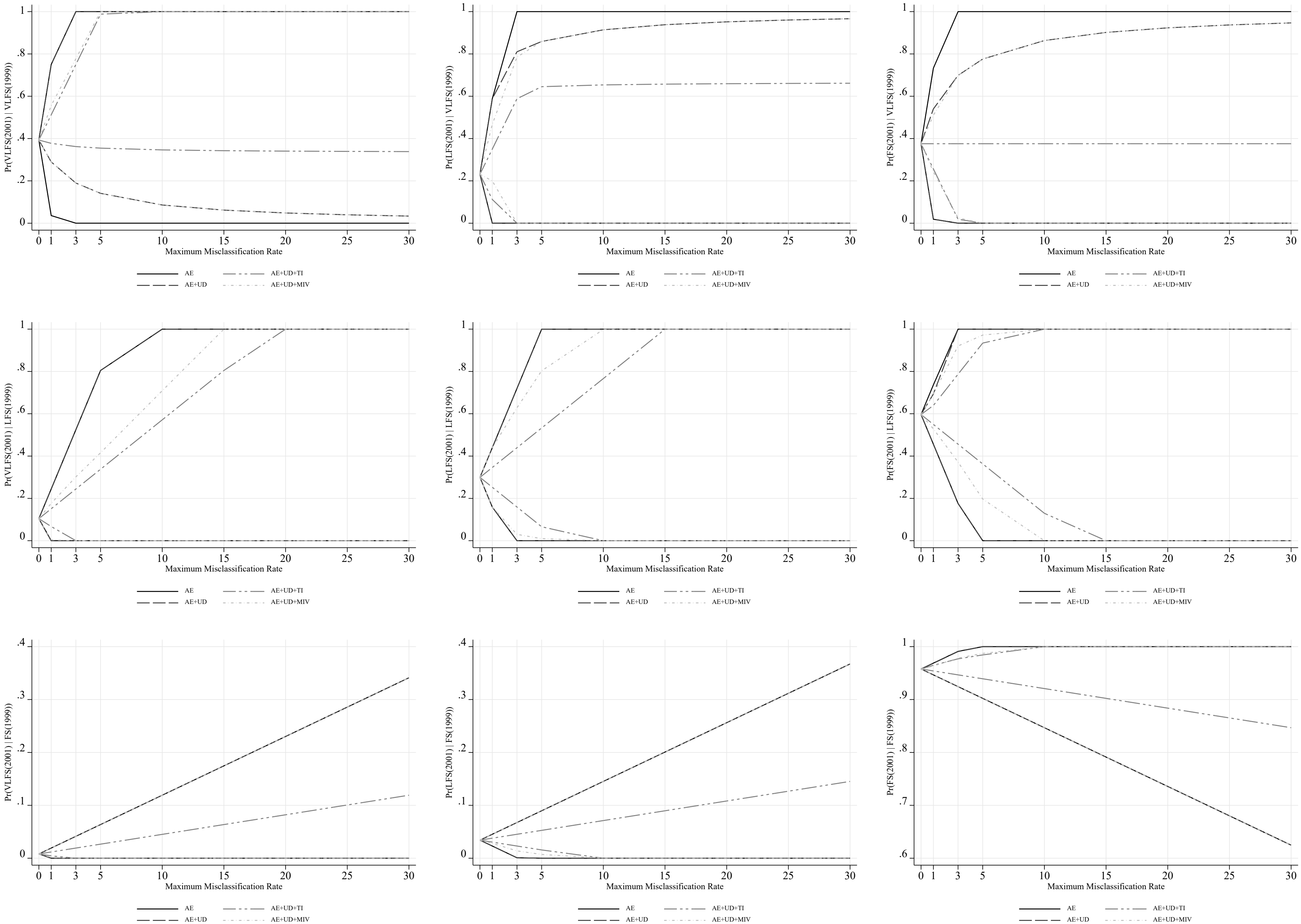

Figure 1. Bounds on Intragenerational Transition Probabilities: 1999 to 2001.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. See text for more details.

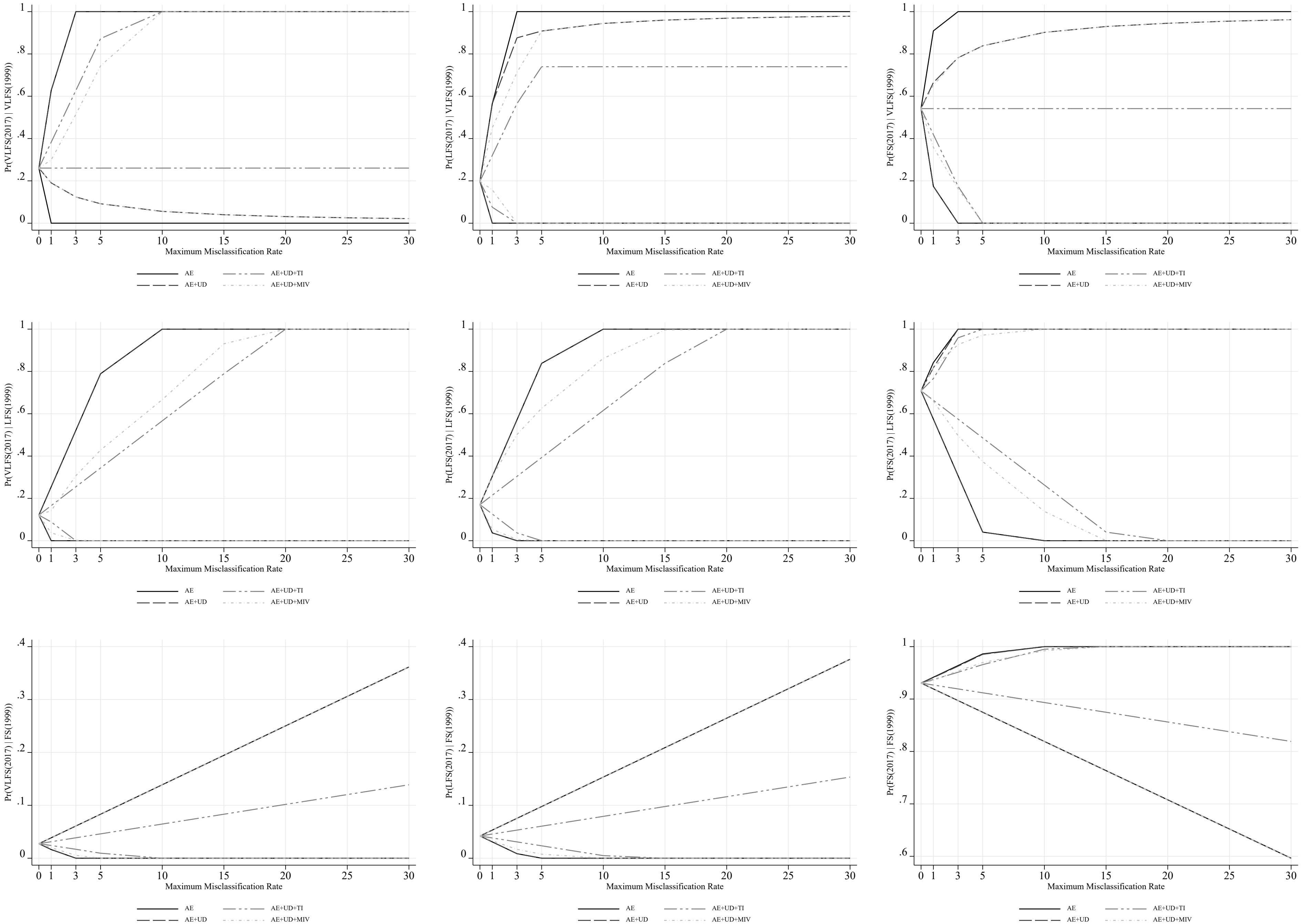

Figure 2. Bounds on Intragenerational Transition Probabilities: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. See text for more details.

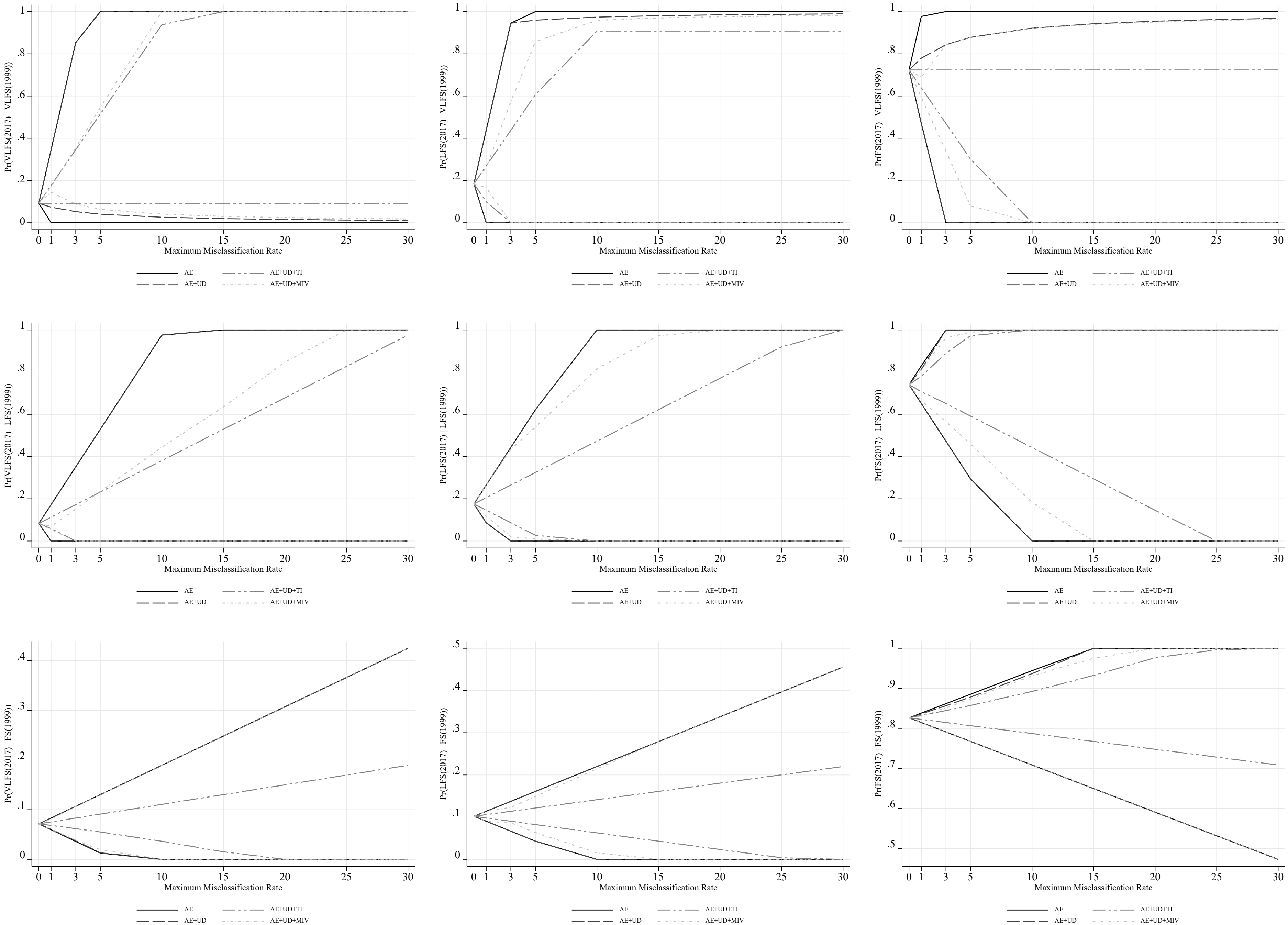

Figure 3. Bounds on Intergenerational Transition Probabilities.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. See text for more details.

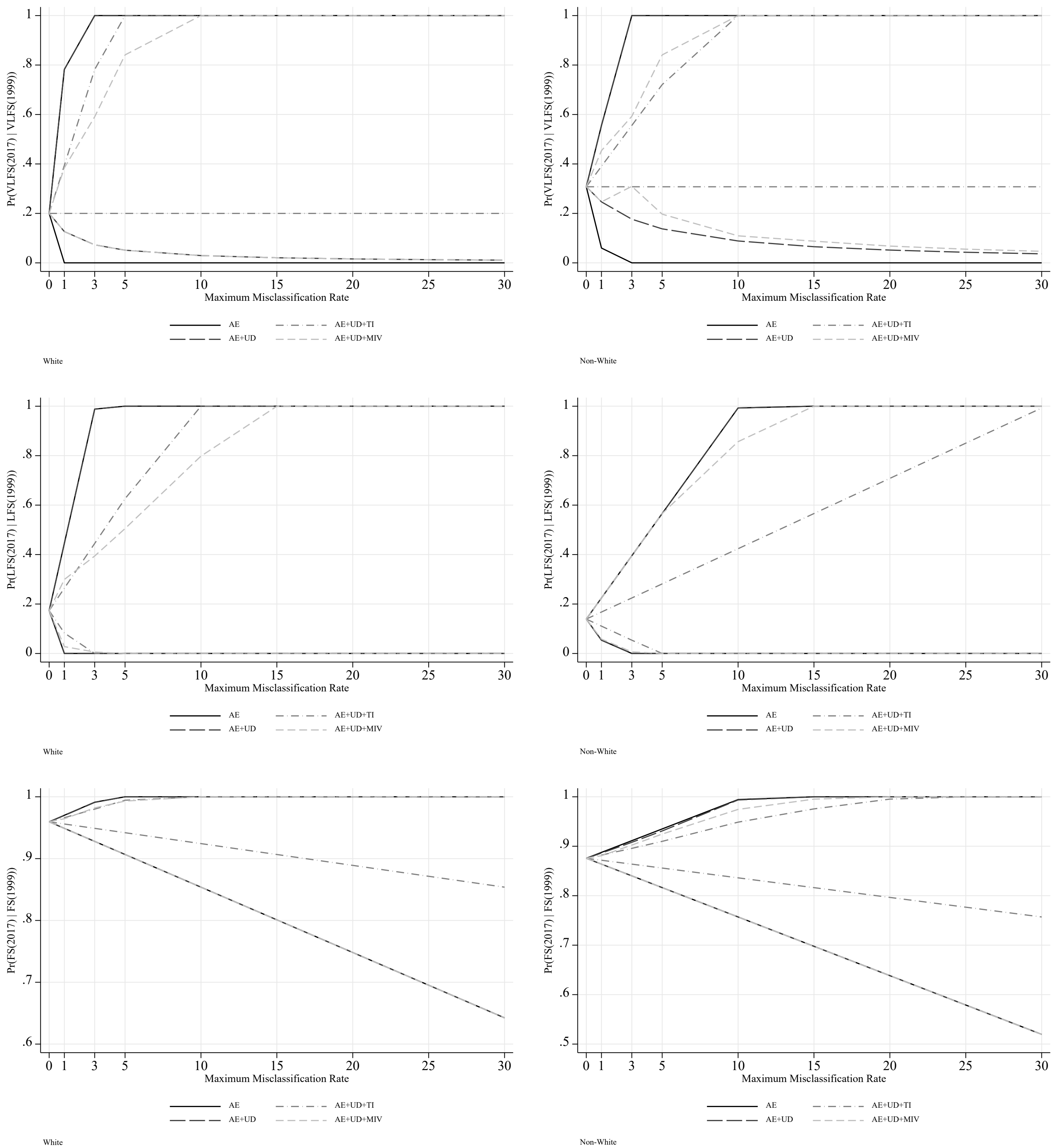

Figure 4. Bounds on Intragenerational Conditional Staying Probabilities by Race: 1999 to 2017

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Left (right) column is for white (non-white) households. See text for more details.

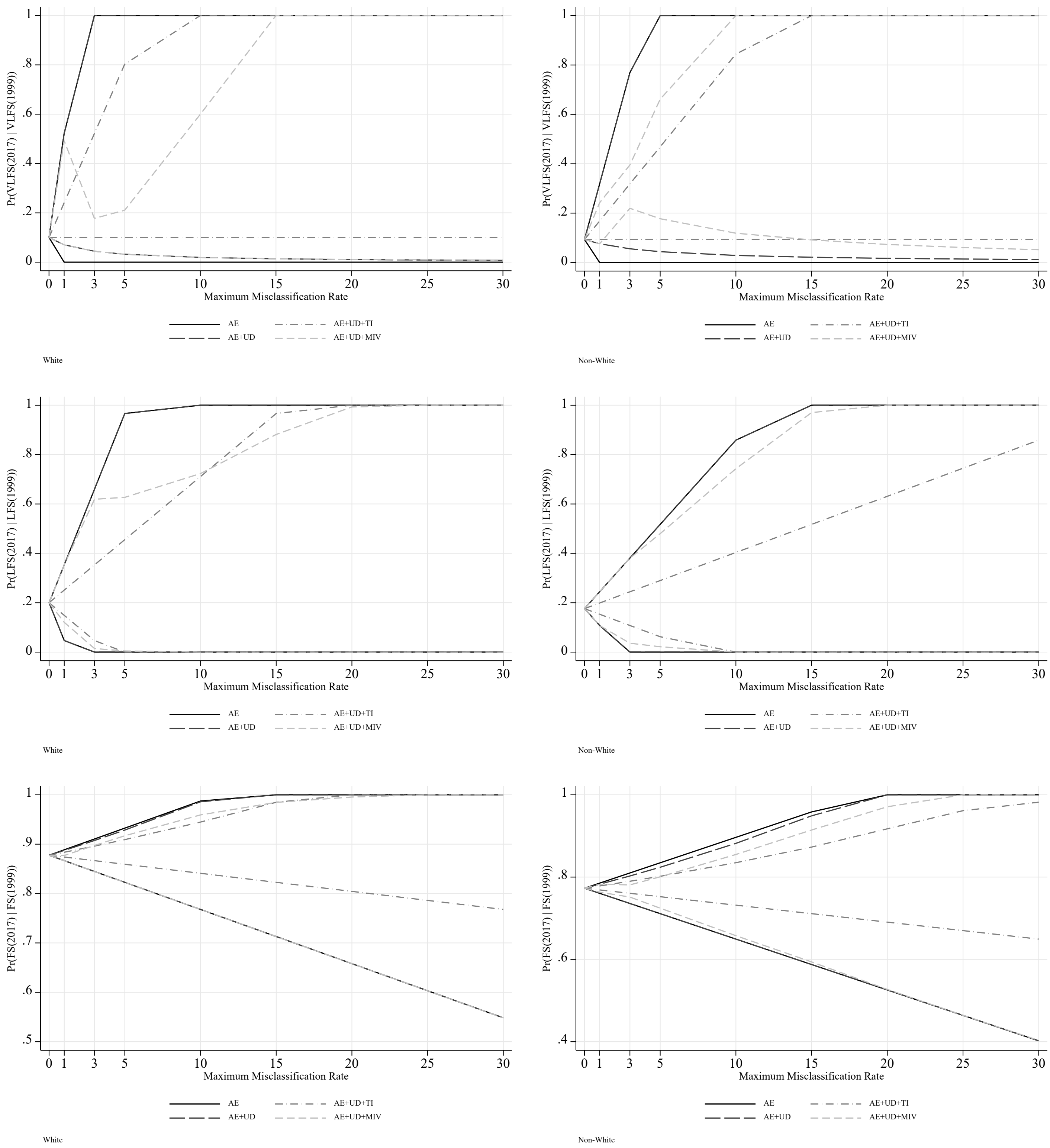

Figure 5. Bounds on Intergenerational Conditional Staying Probabilities by Race: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. Left (right) column is for white (non-white) households. See text for more details.

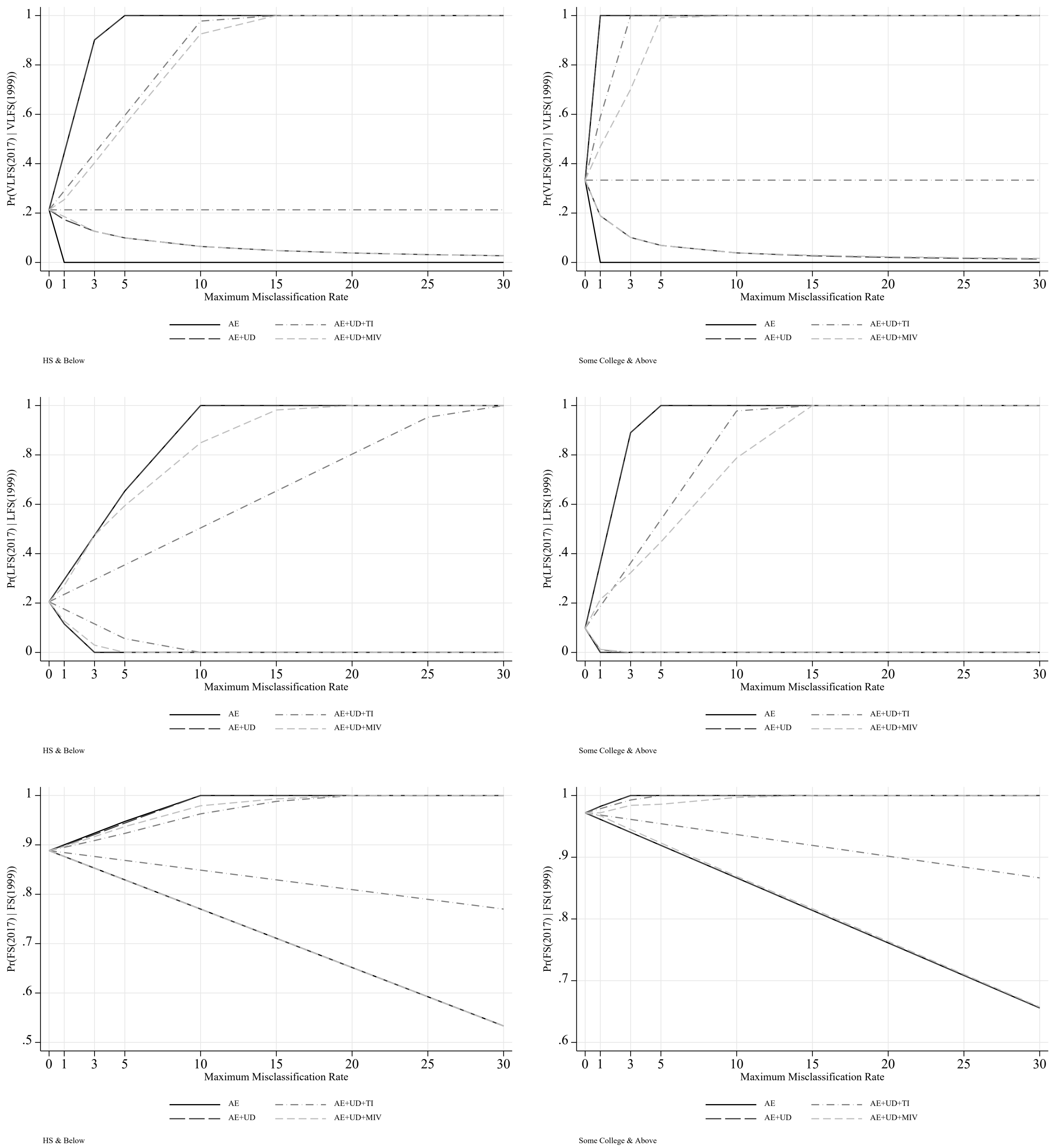

Figure 6. Bounds on Intragenerational Conditional Staying Probabilities by Education: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Left (right) column is for low (high) education households. See text for more details.

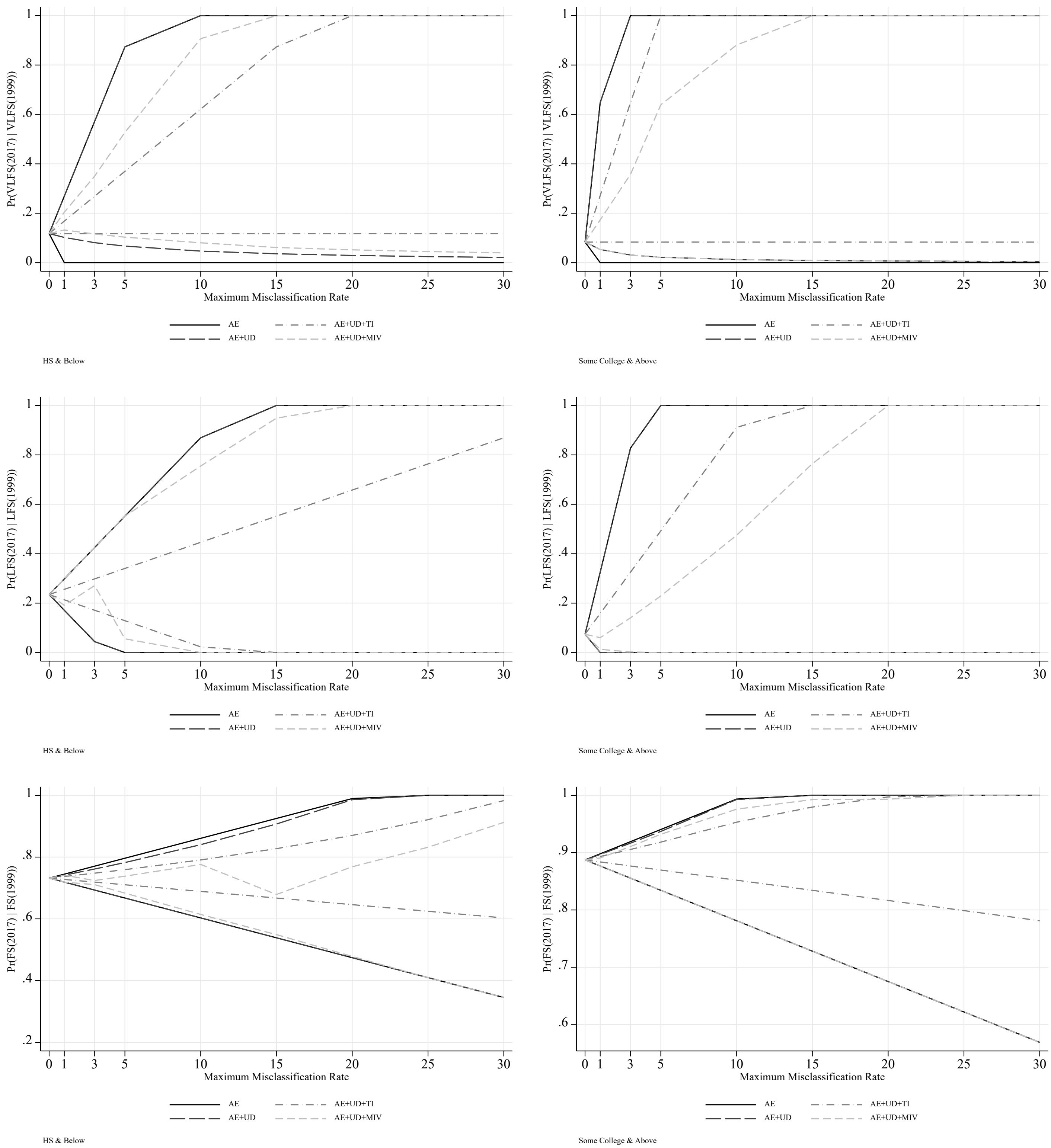

Figure 7. Bounds on Intergenerational Conditional Staying Probabilities by Education: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. Left (right) column is for low (high) education households. See text for more details.

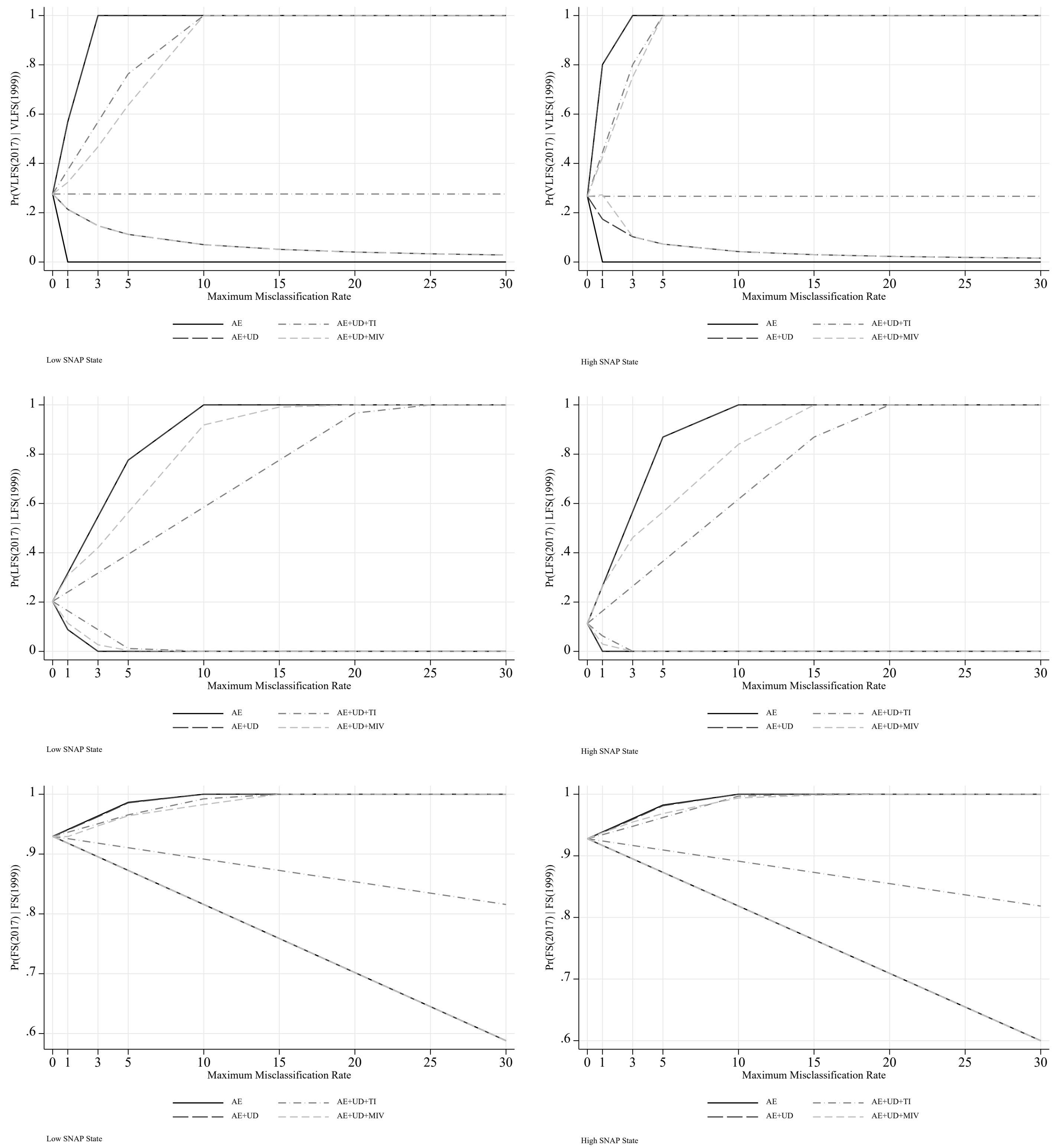

Figure 8. Bounds on Intragenerational Conditional Staying Probabilities by State SNAP Participation: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Left (right) column is for households in low (high) SNAP participation states. See text for more details.

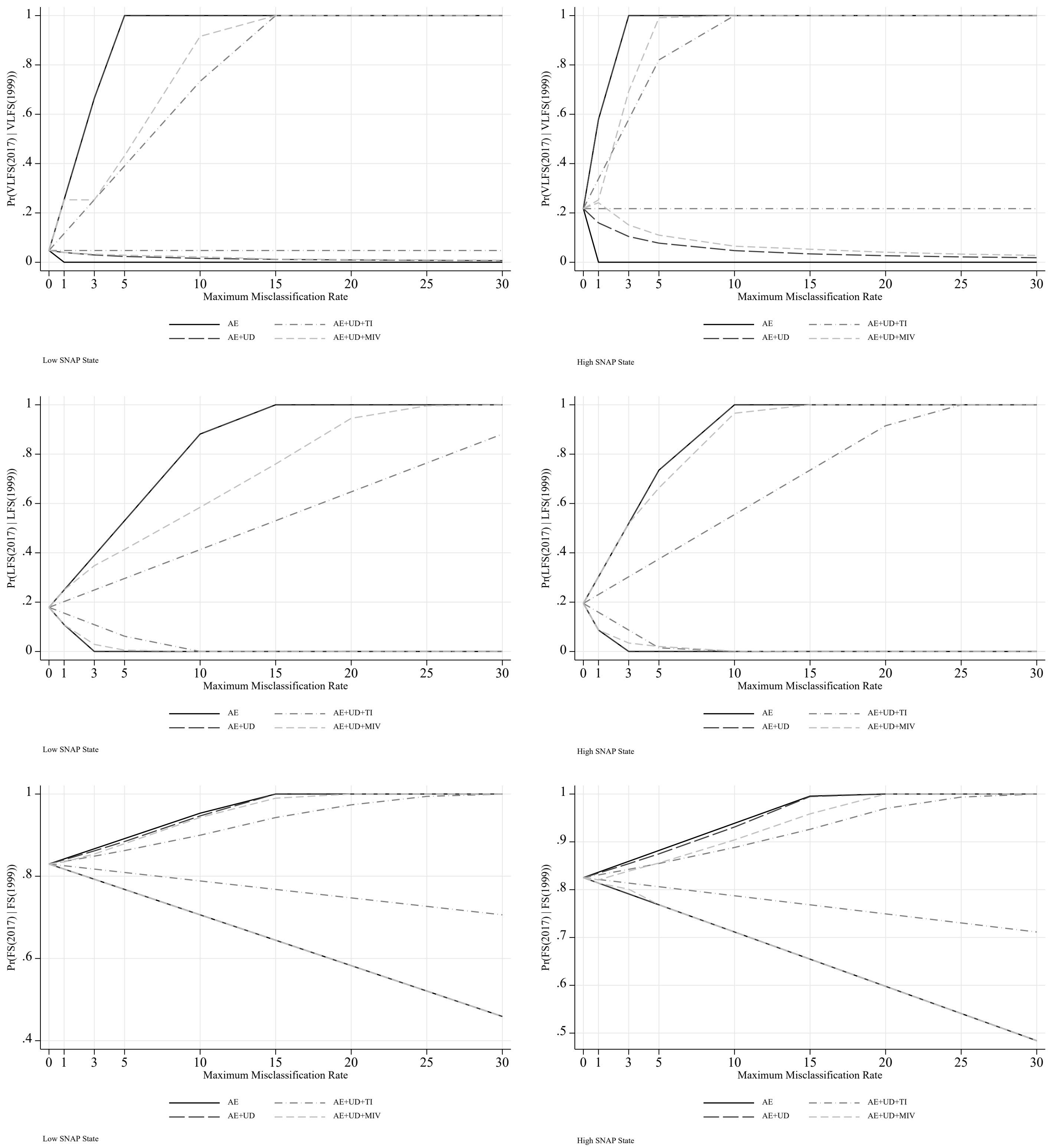

Figure 9. Bounds on Intergenerational Conditional Staying Probabilities by State SNAP Participation: 1999 to 2017.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Sample restricted to adult children who are household heads. Left (right) column is for households in low (high) SNAP participation states. See text for more details.

Supplemental Appendix

Examination of Food Security Dynamics in the Presence of Measurement Error

A Derivation of Bounds

A.1 Misclassification Assumptions

- Baseline case: Assumption 2(i), 2(ii)

$$\theta_{kl}^{k'l'} = \Pr(y_o \in k', y_1 \in l', y_o^* \in k, y_1^* \in l)$$

  - 72 elements
    - General: # elements $= K^2(K^2 - 1)$
  - Under Assumption 2(i)

$$\sum \theta_{kl}^{k'l'} \leq Q$$

  - Under Assumption 2(ii)

$$\begin{aligned}
\sum \theta_{11}^{k'l'} + \sum \theta_{12}^{k'l'} + \sum \theta_{13}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{21}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{23}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{31}^{k'l'} + \sum \theta_{32}^{k'l'} + \sum \theta_{33}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{11}^{k'l'} + \sum \theta_{21}^{k'l'} + \sum \theta_{31}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{12}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{32}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{13}^{k'l'} + \sum \theta_{23}^{k'l'} + \sum \theta_{33}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)}
\end{aligned}$$

- Add Uni-directional assumption
  - Implies

$$\begin{aligned}
\theta_{12}^{11} = \theta_{13}^{11} = \theta_{21}^{11} = \theta_{22}^{11} = \theta_{23}^{11} = \theta_{31}^{11} = \theta_{32}^{11} = \theta_{33}^{11} &= 0 \\
\theta_{13}^{12} = \theta_{21}^{12} = \theta_{22}^{12} = \theta_{23}^{12} = \theta_{31}^{12} = \theta_{32}^{12} = \theta_{33}^{12} &= 0 \\
\theta_{21}^{13} = \theta_{22}^{13} = \theta_{23}^{13} = \theta_{31}^{13} = \theta_{32}^{13} = \theta_{33}^{13} &= 0 \\
\theta_{12}^{21} = \theta_{13}^{21} = \theta_{22}^{21} = \theta_{23}^{21} = \theta_{31}^{21} = \theta_{32}^{21} = \theta_{33}^{21} &= 0 \\
\theta_{13}^{22} = \theta_{23}^{22} = \theta_{31}^{22} = \theta_{32}^{22} = \theta_{33}^{22} &= 0 \\
\theta_{31}^{23} = \theta_{32}^{23} = \theta_{33}^{23} &= 0 \\
\theta_{12}^{31} = \theta_{13}^{31} = \theta_{22}^{31} = \theta_{23}^{31} = \theta_{32}^{31} = \theta_{33}^{31} &= 0 \\
\theta_{13}^{32} = \theta_{23}^{32} = \theta_{33}^{32} &= 0
\end{aligned}$$

  - Now only 27 elements
    - General: # elements $= K^4 + 2K^3 + \sum_k \sum_l [kl - (k+l)(K+1)]$

- Add Temporal Independence assumption

$$
\begin{aligned}
\theta_{kl}^{k'l'} &= \alpha_k^{k'}\beta_l^{l'} \\
\alpha_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) \\
\alpha_k^{k} &= \Pr(y_o \in k, y_o^* \in k) = 1 - \sum_{k' \neq k} \alpha_k^{k'} \\
\beta_l^{l'} &= \Pr(y_1 \in l', y_1^* \in l) \\
\beta_l^{l} &= \Pr(y_1 \in l, y_1^* \in l) = 1 - \sum_{l' \neq l} \beta_l^{l'}
\end{aligned}
$$

  - Now only 12 elements
    - General: # elements = $2K(K-1)$
  - Implies

$$
\begin{array}{lllll}
\theta_{11}^{12} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_1^2 & \theta_{12}^{11} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_2^1 & \theta_{13}^{11} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_3^1 & \theta_{21}^{11} = \alpha_2^1\left(1-\beta_1^2-\beta_1^3\right) & \theta_{22}^{11} = \alpha_2^1\beta_2^1 \\
\theta_{11}^{13} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_1^3 & \theta_{12}^{13} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_2^3 & \theta_{13}^{12} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_3^2 & \theta_{21}^{12} = \alpha_2^1\beta_1^2 & \theta_{22}^{12} = \alpha_2^1\left(1-\beta_2^1-\beta_3^1\right) \\
\theta_{11}^{21} = \alpha_1^2\left(1-\beta_1^2-\beta_1^3\right) & \theta_{12}^{21} = \alpha_1^2\beta_2^1 & \theta_{13}^{21} = \alpha_1^2\beta_3^1 & \theta_{21}^{13} = \alpha_2^1\beta_1^3 & \theta_{22}^{13} = \alpha_2^1\beta_2^3 \\
\theta_{11}^{22} = \alpha_1^2\beta_1^2 & \theta_{12}^{22} = \alpha_1^2\left(1-\beta_2^1-\beta_2^3\right) & \theta_{13}^{22} = \alpha_1^2\beta_3^2 & \theta_{21}^{22} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_1^2 & \theta_{22}^{21} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_2^1 \\
\theta_{11}^{23} = \alpha_1^2\beta_1^3 & \theta_{12}^{23} = \alpha_1^2\beta_2^3 & \theta_{13}^{23} = \alpha_1^2\left(1-\beta_3^1-\beta_3^3\right) & \theta_{21}^{23} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_1^3 & \theta_{22}^{23} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_2^3 \\
\theta_{11}^{31} = \alpha_1^3\left(1-\beta_1^2-\beta_1^3\right) & \theta_{12}^{31} = \alpha_1^3\beta_2^1 & \theta_{13}^{31} = \alpha_1^3\beta_3^1 & \theta_{21}^{31} = \alpha_2^3\left(1-\beta_1^2-\beta_1^3\right) & \theta_{22}^{31} = \alpha_2^3\beta_2^1 \\
\theta_{11}^{32} = \alpha_1^3\beta_1^2 & \theta_{12}^{32} = \alpha_1^3\left(1-\beta_2^1-\beta_2^3\right) & \theta_{13}^{32} = \alpha_1^3\beta_3^2 & \theta_{21}^{32} = \alpha_2^3\beta_1^2 & \theta_{22}^{32} = \alpha_2^3\left(1-\beta_2^1-\beta_2^3\right) \\
\theta_{11}^{33} = \alpha_1^3\beta_1^3 & \theta_{12}^{33} = \alpha_1^3\beta_2^3 & \theta_{13}^{33} = \alpha_1^3\left(1-\beta_3^1-\beta_3^2\right) & \theta_{21}^{33} = \alpha_2^1\beta_1^3 & \theta_{22}^{33} = \alpha_2^3\beta_2^3
\end{array}
$$

$$
\begin{array}{llll}
\theta_{23}^{11} = \alpha_2^1\beta_3^1 & \theta_{31}^{11} = \alpha_3^1\left(1-\beta_1^2-\beta_1^3\right) & \theta_{32}^{11} = \alpha_3^1\beta_2^1 & \theta_{33}^{11} = \alpha_3^1\beta_3^1 \\
\theta_{23}^{12} = \alpha_2^1\beta_3^2 & \theta_{31}^{12} = \alpha_3^1\beta_1^2 & \theta_{32}^{12} = \alpha_3^1\left(1-\beta_2^1-\beta_2^3\right) & \theta_{33}^{12} = \alpha_3^1\beta_3^2 \\
\theta_{23}^{13} = \alpha_2^1\left(1-\beta_3^1-\beta_3^2\right) & \theta_{31}^{13} = \alpha_3^1\beta_1^3 & \theta_{32}^{13} = \alpha_3^1\beta_2^3 & \theta_{33}^{13} = \alpha_3^1\left(1-\beta_3^1-\beta_3^3\right) \\
\theta_{23}^{21} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_3^1 & \theta_{31}^{21} = \alpha_3^2\left(1-\beta_1^2-\beta_1^3\right) & \theta_{32}^{21} = \alpha_3^2\beta_2^1 & \theta_{33}^{21} = \alpha_3^2\beta_3^1 \\
\theta_{23}^{22} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_3^2 & \theta_{31}^{22} = \alpha_3^2\beta_1^2 & \theta_{32}^{22} = \alpha_3^2\left(1-\beta_2^1-\beta_2^3\right) & \theta_{33}^{22} = \alpha_3^2\beta_3^2 \\
\theta_{23}^{31} = \alpha_2^3\beta_3^1 & \theta_{31}^{23} = \alpha_3^2\beta_1^3 & \theta_{32}^{23} = \alpha_3^2\beta_2^3 & \theta_{33}^{23} = \alpha_3^2\left(1-\beta_3^1-\beta_3^3\right) \\
\theta_{23}^{32} = \alpha_2^3\beta_3^2 & \theta_{31}^{32} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_1^2 & \theta_{32}^{31} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_2^1 & \theta_{33}^{31} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_3^1 \\
\theta_{23}^{33} = \alpha_2^3\left(1-\beta_3^1-\beta_3^2\right) & \theta_{11}^{33} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_1^3 & \theta_{32}^{33} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_2^3 & \theta_{33}^{32} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_3^2
\end{array}
$$

    - Under Assumption 2(i)

$$
\begin{aligned}
\sum \theta_{kl}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad + \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad + \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad + \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad + \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad + \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad + \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&\quad + \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&\quad + \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&= \left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right) \\
&\qquad +3\left(\alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2\right) \leq Q \\
&\Rightarrow \alpha_1^2,\alpha_1^3,\alpha_2^1,\alpha_2^3,\alpha_3^1,\alpha_3^2,\beta_1^2,\beta_1^3,\beta_2^1,\beta_2^3,\beta_3^1,\beta_3^2 \leq Q/3 \\
&\Rightarrow \alpha,\beta \leq Q/K \qquad \text{(generally)}
\end{aligned}
$$

∗ Under Assumption 2(ii)

$$\begin{aligned}
\sum \theta_{11}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{12}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{13}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{21}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{22}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{23}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{31}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
\sum \theta_{32}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
\sum \theta_{33}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_3^1+\alpha_3^2\right)
\end{aligned}$$

and the following restrictions must hold

$$\begin{aligned}
\sum \theta_{11}^{k'l'}+\sum \theta_{12}^{k'l'}+\sum \theta_{13}^{k'l'} &= 3\left(\alpha_1^2+\alpha_1^3\right)+\left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{21}^{k'l'}+\sum \theta_{22}^{k'l'}+\sum \theta_{23}^{k'l'} &= 3\left(\alpha_2^1+\alpha_2^3\right)+\left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{31}^{k'l'}+\sum \theta_{32}^{k'l'}+\sum \theta_{33}^{k'l'} &= 3\left(\alpha_3^1+\alpha_3^2\right)+\left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{11}^{k'l'}+\sum \theta_{21}^{k'l'}+\sum \theta_{31}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)\leq Q/3 \\
\sum \theta_{12}^{k'l'}+\sum \theta_{22}^{k'l'}+\sum \theta_{32}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)\leq Q/3 \\
\sum \theta_{13}^{k'l'}+\sum \theta_{23}^{k'l'}+\sum \theta_{33}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)\leq Q/3
\end{aligned}$$

which imply

$$\begin{aligned}
&\Rightarrow \quad \alpha_1^2,\alpha_1^3,\alpha_2^1,\alpha_2^3,\alpha_3^1,\alpha_3^2,\beta_1^2,\beta_1^3,\beta_2^1,\beta_2^3,\beta_3^1,\beta_3^2\leq Q/9 \\
&\Rightarrow \quad \left(\left\{\alpha_1^2,\alpha_1^3\right\}+\left\{\alpha_2^1,\alpha_2^3\right\}+\left\{\alpha_3^1,\alpha_3^2\right\}\right),\left(\left\{\beta_1^2,\beta_1^3\right\}+\left\{\beta_2^1,\beta_2^3\right\}+\left\{\beta_3^1,\beta_3^2\right\}\right)\leq Q/3 \\
&\Rightarrow \quad \alpha,\beta\leq Q/K^2 \qquad \text{(generally)} \\
&\Rightarrow \quad \textstyle\sum_k \left\{\alpha_k^{k'}\right\},\sum_k \left\{\beta_k^{k'}\right\}\leq Q/K \qquad \text{(generally)}
\end{aligned}$$

– Add Uni-directional assumption
  ∗ Implies

$$\alpha_2^1=\beta_2^1=\alpha_3^1=\beta_3^1=\alpha_3^2=\beta_3^2=0$$

  ∗ Now only 6 elements
  ∗ General: # elements = $K(K-1)$

- Add Temporal Invariance assumption

$$\begin{aligned}
\theta_{kl}^{k'l'} &= \theta_k^{k'}\theta_l^{l'} \\
\theta_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) = \Pr(y_1 \in k', y_1^* \in k) \\
\theta_k^{k} &= \Pr(y_o \in k, y_o^* \in k) = \Pr(y_1 \in k, y_1^* \in k) = 1 - \theta_k^{k'}
\end{aligned}$$

  - Now only 6 elements
    - General: # elements $= K(K-1)$
  - Implies

$$\begin{array}{llll}
\theta_{11}^{12} = \theta_1^2\left(1-\theta_1^2\right) & \theta_{12}^{11} = \theta_2^1\left(1-\theta_1^2\right) & \theta_{21}^{11} = \theta_2^1\left(1-\theta_1^2\right) & \theta_{22}^{11} = \left(\theta_2^1\right)^2 \\
\theta_{11}^{21} = \theta_1^2\left(1-\theta_1^2\right) & \theta_{12}^{21} = \theta_1^2\theta_2^1 & \theta_{21}^{12} = \theta_2^1\theta_1^2 & \theta_{22}^{12} = \theta_2^1\left(1-\theta_2^1\right) \\
\theta_{11}^{22} = \left(\theta_1^2\right)^2 & \theta_{12}^{22} = \theta_1^2\left(1-\theta_2^1\right) & \theta_{21}^{22} = \theta_1^2\left(1-\theta_2^1\right) & \theta_{22}^{21} = \theta_2^1\left(1-\theta_2^1\right)
\end{array}$$

    - Under Assumption 2(i) (solution: set all θs but one to zero, solve using quadratic formula)

$$\begin{aligned}
\sum \theta_{kl}^{k'l'} &= \left(6-\theta_1^2-\theta_1^3-\theta_2^1-\theta_2^3-\theta_3^1-\theta_3^2\right)\left(\theta_1^2+\theta_1^3+\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right) \le Q \\
&\Rightarrow \theta_1^2, \theta_1^3, \theta_2^1, \theta_2^3, \theta_3^1, \theta_3^2 \le 3 - \sqrt{9-Q} \\
&\Rightarrow \theta \le K - \sqrt{K^2 - Q} \qquad \text{(generally)}
\end{aligned}$$

    - Under Assumption 2(ii) (solution: set all θs but one to zero, solve using quadratic formula)

$$\begin{aligned}
\sum \theta_{11}^{k'l'} + \sum \theta_{12}^{k'l'} + \sum \theta_{13}^{k'l'} &= 4\left(\theta_1^2+\theta_1^3\right) + \left(1-\theta_1^2-\theta_1^3\right)\left(\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right) - \left(\theta_1^2+\theta_1^3\right)^2 \le Q/3 \\
\sum \theta_{21}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{23}^{k'l'} &= 4\left(\theta_2^1+\theta_2^3\right) + \left(1-\theta_2^1-\theta_2^3\right)\left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right) - \left(\theta_2^1+\theta_2^3\right)^2 \le Q/3 \\
\sum \theta_{31}^{k'l'} + \sum \theta_{32}^{k'l'} + \sum \theta_{33}^{k'l'} &= 4\left(\theta_3^1+\theta_3^2\right) + \left(1-\theta_3^1-\theta_3^2\right)\left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right) - \left(\theta_3^1+\theta_3^2\right)^2 \le Q/3 \\
\sum \theta_{11}^{k'l'} + \sum \theta_{21}^{k'l'} + \sum \theta_{31}^{k'l'} &= 4\left(\theta_1^2+\theta_1^3\right) + \left(\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right)\left(1-\theta_1^2-\theta_1^3\right) - \left(\theta_1^2+\theta_1^3\right)^2 \le Q/3 \\
\sum \theta_{12}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{32}^{k'l'} &= 4\left(\theta_2^1+\theta_2^3\right) + \left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right) - \left(\theta_2^1+\theta_2^3\right)^2 \le Q/3 \\
\sum \theta_{13}^{k'l'} + \sum \theta_{23}^{k'l'} + \sum \theta_{33}^{k'l'} &= 4\left(\theta_3^1+\theta_3^2\right) + \left(\theta_1^2+\theta_1^3+\theta_2^1+\theta_2^3\right)\left(1-\theta_3^1-\theta_3^2\right) - \left(\theta_3^1+\theta_3^2\right)^2 \le Q/3
\end{aligned}$$

    which imply

$$\begin{aligned}
&\Rightarrow \theta_1^2, \theta_1^3, \theta_2^1, \theta_2^3, \theta_3^1, \theta_3^2 \le \left(4 - \sqrt{16 - 4Q/3}\right)/2 \\
&\Rightarrow \theta \le \left(K + 1 - \sqrt{(K+1)^2 - 4Q/K}\right)/2 \qquad \text{(generally)}
\end{aligned}$$

  - Add Uni-directional assumption
    - Implies

$$\theta_2^1 = \theta_3^1 = \theta_3^2 = 0$$

    - Now only 3 elements
    - General: # elements $= K(K-1)/2$

A.1.1 p_{11}^*

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}} + \underset{\bullet}{\theta_{11}^{13}} + \theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33}\right]}^{Q_{1,11}} - \overbrace{\left[\underset{\bullet}{\theta_{12}^{11}} + \underset{\bullet}{\theta_{13}^{11}} + \theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11}\right]}^{Q_{2,11}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \underset{\bullet}{\theta_{12}^{21}} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{12}^{31}} + \underset{\bullet}{\theta_{12}^{32}} + \underset{\bullet}{\theta_{12}^{33}} + \underset{\bullet}{\theta_{13}^{21}} + \underset{\bullet}{\theta_{13}^{22}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{31}} + \underset{\bullet}{\theta_{13}^{32}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}} - \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \underset{\bullet}{\theta_{21}^{12}} + \underset{\bullet}{\theta_{22}^{12}} + \underset{\bullet}{\theta_{23}^{12}} + \underset{\bullet}{\theta_{21}^{13}} + \underset{\bullet}{\theta_{22}^{13}} + \underset{\bullet}{\theta_{23}^{13}} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \underset{\bullet}{\theta_{31}^{12}} + \underset{\bullet}{\theta_{32}^{12}} + \underset{\bullet}{\theta_{33}^{12}} + \underset{\bullet}{\theta_{31}^{13}} + \underset{\bullet}{\theta_{32}^{13}} + \underset{\bullet}{\theta_{33}^{13}}\right]}_{Q_{4,1}}}$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{11} = \frac{r_{11} - \widetilde{Q}}{p_1} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{11} = \frac{r_{11} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}} + \underset{\bullet}{\theta_{11}^{13}} + \theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33}\right]}^{Q_{1,11}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{12}^{32}} + \underset{\bullet}{\theta_{12}^{33}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}^u}}$$

- Yields

$$LB_{11}^u = \frac{r_{11}}{p_1 + \widetilde{\widetilde{Q}}} \geq 0 \qquad \widetilde{\widetilde{Q}} = \min\left\{1 - p_1, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{11}^u = \frac{r_{11} + \widetilde{Q}}{p_1} \leq 1 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{11}^{*} = \frac{r_{11} + \overbrace{\left[\alpha_1^1\beta_1^2 + \alpha_1^1\beta_1^3 + \alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3\right]}^{Q_{1,11}} - \overbrace{\left[\alpha_1^1\beta_2^1 + \alpha_1^1\beta_3^1 + \alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1\right]}^{Q_{2,11}}}{p_1 + \underbrace{\left[\alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3 + \alpha_1^2\beta_3^1 + \alpha_1^2\beta_3^2 + \alpha_1^2\beta_3^3 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_3^3\right]}_{Q_{3,1}} - \underbrace{\left[\alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_2^1\beta_1^3 + \alpha_2^1\beta_2^3 + \alpha_2^1\beta_3^3 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_1^3 + \alpha_3^1\beta_2^3 + \alpha_3^1\beta_3^3\right]}_{Q_{4,1}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,11} &= \left(\alpha_1^2 + \alpha_1^3\right) + \left(\beta_1^2 + \beta_1^3\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= 2\left(\theta_1^2 + \theta_1^3\right) - \left(\theta_1^2 + \theta_1^3\right)^2 && \text{(TIV)} \\
Q_{2,11} &= \left(\alpha_2^1 + \alpha_3^1\right)\left(1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3\right) + \left(\beta_2^1 + \beta_3^1\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= 2\left(\theta_2^1 + \theta_3^1\right)\left(1 - \theta_1^2 - \theta_1^3\right) + \left(\theta_2^1 + \theta_3^1\right)^2 && \text{(TIV)} \\
Q_{3,1} &= 3\left(\alpha_1^2 + \alpha_1^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^2 + \theta_1^3\right) && \text{(TIV)} \\
Q_{4,1} &= 3\left(\alpha_2^1 + \alpha_3^1\right) && \text{(TI)} \\
&= 3\left(\theta_2^1 + \theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{11}^{*}=\frac{r_{11}+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)\left(1+\alpha_2^1+\alpha_3^1-\alpha_1^2-\alpha_1^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

 - Yields

$$LB_{11}^{TI} = \min\left\{\frac{r_{11}-\ddot{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}},\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{11},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{11}^{TI} = \max\left\{\frac{r_{11}+\widetilde{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}},\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1$$

$$\widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{11},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

 - Proof:
   1. $\ddot{Q}$ in $\frac{r_{11}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
   2. $\widehat{Q}$ in $\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
   3. Evaluate $\partial\left(\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_1+3\widetilde{\widetilde{Q}}\right)-3\left(r_{11}+\widetilde{\widetilde{Q}}\right)\right)\\ &= sgn\left(p_1-3r_{11}\right)\end{aligned}$$

   4. Evaluate $\partial\left(\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{11}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{11}-p_1\right)\end{aligned}$$

 - Adding the uni-directional assumption

$$p_{11}^{*}=\frac{r_{11}+\left(\alpha_1^2+\alpha_1^3\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_1^2-\alpha_1^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

   * Yields

$$LB_{11}^{TI,u} = \frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\geq 0 \qquad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{11}<p_1/3\\ \min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

$$UB_{11}^{TI,u} = \max\left\{\frac{r_{11}+\widetilde{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

   * Proof: Same as above.

- Under Temporal Invariance

$$p_{11}^{*} = \frac{r_{11} + 2\left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right) - \left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)^2}{p_1 + 3\left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)}$$

  - Yields

$$\begin{aligned}
LB_{11}^{TIV} &= \min\left\{\frac{r_{11} + 2\widehat{Q} - \widehat{Q}^2}{p_1 + 3\widehat{Q}}, \frac{r_{11} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \widehat{Q} = \min\left\{(1-p_1)/3, \widetilde{Q}\right\}, \\
& \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\
UB_{11}^{TIV} &= \max\left\{\frac{r_{11} + 2\widehat{Q} - \widehat{Q}^2}{p_1 + 3\widehat{Q}}, \frac{r_{11} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \widehat{Q} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (1-p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} < 2p_1/3 \\ \min\left\{\frac{(2/3)p_1 - \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(-2-2\tilde{\tilde{Q}}\right)\left(p_1-3\tilde{\tilde{Q}}\right)+3\left(r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\right)\right) \\
&= sgn\left(-(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=0} &= sgn\left(-(2/3)p_1+r_{11}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=1} &= sgn\left(-(4/3)p_1+1+r_{11}\right) \gtrless 0
\end{aligned}$$

2. Ensure $r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0$

$$\begin{aligned}
& r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}}^2+2\tilde{\tilde{Q}}-r_{11} \leq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq \frac{-2+\sqrt{4+4r_{11}}}{2} \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq -1+\sqrt{1+r_{11}}
\end{aligned}$$

3. Minimize $\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{11} < 2p_1/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}=0 \\
\Rightarrow \tilde{\tilde{Q}}^* &= \frac{(2/3)p_1+\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{11} > 2p_1/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}=0 \\
\Rightarrow \tilde{\tilde{Q}}^* &= \frac{(2/3)p_1-\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]} = .$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$
\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_1+3\widehat{Q}\right)-3\left(r_{11}+2\widehat{Q}-\widehat{Q}^2\right)\right) \\
&= sgn\left((2/3)p_1\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{11}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_1-r_{11}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{11}\right) < 0
\end{aligned}
$$

6. Maximize $\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{11} < 2p_1/3$

$$
\begin{aligned}
\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widetilde{\widehat{Q}}} &\propto (2/3)p_1\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{11}=0 \\
&\Rightarrow \widetilde{\widehat{Q}}^* = \frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}
$$

7. Minimize $\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}} \Rightarrow \widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{11}-2\widetilde{\widehat{Q}}-\widetilde{\widehat{Q}}^2}{p_1-3\widetilde{\widehat{Q}}} < \frac{r_{11}}{p_1}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p_{11}^{*} = \frac{r_{11} + 2\theta_1^2 - \left(\theta_1^2\right)^2}{p_1 + 2\theta_1^2}$$

∗ Yields

$$\begin{aligned}
LB_{11}^{TIV,u} &= \min\left\{\frac{r_{11}}{p_1}, \frac{r_{11} + 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
&\quad \widetilde{\widetilde{Q}} = \min\left\{(1-p_1)/3, \widetilde{Q}\right\}, \\
&\quad \widetilde{Q} = \begin{cases} 3 - \sqrt{9-Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\
UB_{11}^{TIV,u} &= \frac{r_{11} + 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}} \leq 1 \\
&\quad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (1-p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
&\quad \widetilde{Q} = \begin{cases} 3 - \sqrt{9-Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}
\end{aligned}$$

∗ Proof: Same as above.

A.1.2 p^*_{12}

$$p^*_{12} = \frac{r_{12} + \overbrace{\left[\underset{\bullet}{\theta^{11}_{12}} + \underset{\bullet}{\theta^{13}_{12}} + \theta^{21}_{12} + \theta^{22}_{12} + \theta^{23}_{12} + \theta^{31}_{12} + \theta^{32}_{12} + \theta^{33}_{12}\right]}^{Q_{1,12}} - \overbrace{\left[\underset{\bullet}{\theta^{12}_{11}} + \underset{\bullet}{\theta^{12}_{13}} + \theta^{12}_{21} + \theta^{12}_{22} + \theta^{12}_{23} + \theta^{12}_{31} + \theta^{12}_{32} + \theta^{12}_{33}\right]}^{Q_{2,12}}}{\begin{array}{l} p_1 + \underbrace{\left[\theta^{21}_{11} + \theta^{22}_{11} + \theta^{23}_{11} + \theta^{31}_{11} + \theta^{32}_{11} + \theta^{33}_{11} + \underset{\bullet}{\theta^{21}_{12}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{31}_{12}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{21}_{13}} + \underset{\bullet}{\theta^{22}_{13}} + \underset{\bullet}{\theta^{23}_{13}} + \underset{\bullet}{\theta^{31}_{13}} + \underset{\bullet}{\theta^{32}_{13}} + \underset{\bullet}{\theta^{33}_{13}}\right]}_{Q_{3,1}} \\ \quad - \underbrace{\left[\theta^{11}_{21} + \theta^{11}_{22} + \theta^{11}_{23} + \underset{\bullet}{\theta^{12}_{21}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{21}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{13}_{23}} + \theta^{11}_{31} + \theta^{11}_{32} + \theta^{11}_{33} + \underset{\bullet}{\theta^{12}_{31}} + \underset{\bullet}{\theta^{12}_{32}} + \underset{\bullet}{\theta^{12}_{33}} + \underset{\bullet}{\theta^{13}_{31}} + \underset{\bullet}{\theta^{13}_{32}} + \underset{\bullet}{\theta^{13}_{33}}\right]}_{Q_{4,1}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{12} &= \frac{r_{12} - \widetilde{Q}}{p_1} \geq 0 \qquad & \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{12} &= \frac{r_{12} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 \qquad & \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{12} = \frac{r_{12} + \overbrace{\left[\underset{\bullet}{\theta^{13}_{12}} + \theta^{22}_{12} + \theta^{23}_{12} + \theta^{32}_{12} + \theta^{33}_{12}\right]}^{Q^u_{1,12}} - \overbrace{\left[\underset{\bullet}{\theta^{12}_{11}}\right]}^{Q^u_{2,12}}}{p_1 + \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{31}_{11}} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{33}_{11}} + \theta^{22}_{12} + \theta^{23}_{12} + \theta^{32}_{12} + \theta^{33}_{12} + \underset{\bullet}{\theta^{23}_{13}} + \underset{\bullet}{\theta^{33}_{13}}\right]}_{Q^u_{3,1}}}$$

- Yields

$$\begin{aligned} LB^u_{12} &= \min\left\{\frac{r_{12} - \widetilde{Q}}{p_1}, \frac{r_{12}}{p_1 + \widetilde{\widetilde{Q}}}\right\} \geq 0 \qquad & \widetilde{\widetilde{Q}} &= \min\{1 - p_1, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB^u_{12} &= \frac{r_{12} + \widetilde{Q}}{p_1} \leq 1 \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{12}^{*} = \frac{r_{12} + \overbrace{\left[\alpha_1^1\beta_2^1 + \alpha_1^1\beta_2^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3\right]}^{Q_{1,12}} - \overbrace{\left[\alpha_1^1\beta_1^2 + \alpha_1^1\beta_3^2 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2\right]}^{Q_{2,12}}}{p_1 + \underbrace{\left[\alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3 + \alpha_1^2\beta_3^1 + \alpha_1^2\beta_3^2 + \alpha_1^2\beta_3^3 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_3^3\right]}_{Q_{3,1}} - \underbrace{\left[\alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_2^1\beta_1^3 + \alpha_2^1\beta_2^3 + \alpha_2^1\beta_3^3 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_1^3 + \alpha_3^1\beta_2^3 + \alpha_3^1\beta_3^3\right]}_{Q_{4,1}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,12} &= \left(\alpha_1^2 + \alpha_1^3\right) + \left(\beta_2^1 + \beta_2^3\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= \left(\theta_1^2 + \theta_1^3\right) + \left(\theta_2^1 + \theta_2^3\right)\left(1 - \theta_1^2 - \theta_1^3\right) && \text{(TIV)} \\
Q_{2,12} &= \left(\alpha_2^1 + \alpha_3^1\right)\left(1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3\right) + \left(\beta_1^2 + \beta_3^2\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= \left(\theta_2^1 + \theta_3^1\right)\left(1 + \theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3\right) + \left(\theta_1^2 + \theta_3^2\right)\left(1 - \theta_1^2 - \theta_1^3\right) && \text{(TIV)} \\
Q_{3,1} &= 3\left(\alpha_1^2 + \alpha_1^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^2 + \theta_1^3\right) && \text{(TIV)} \\
Q_{4,1} &= 3\left(\alpha_2^1 + \alpha_3^1\right) && \text{(TI)} \\
&= 3\left(\theta_2^1 + \theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{12}^{*}=\frac{r_{12}+\left(\beta_2^1+\beta_2^3-\beta_1^2-\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

  - Yields

$$LB_{12}^{TI} = \min\left\{\frac{r_{12}-\ddot{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\tilde{Q}\right\},\ \widehat{Q}<\min\left\{r_{12},p_1/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{12}^{TI} = \max\left\{\frac{r_{12}+\tilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\tilde{Q}\right\},\ \widehat{Q}<\min\left\{r_{12},p_1/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{12}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_1^2,\beta_3^2=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$sgn\left(\frac{\partial\left(\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) = sgn\left(\left(p_1+3\tilde{\tilde{Q}}\right)-3\left(r_{12}+\tilde{\tilde{Q}}\right)\right) = sgn\left(p_1-3r_{12}\right)$$

    4. Evaluate $\partial\left(\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$sgn\left(\frac{\partial\left(\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) = sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{12}-\widehat{Q}\right)\right) = sgn\left(3r_{12}-p_1\right)$$

    Since both derivatives can take either sign, it is possible either could be the LB, UB.

  - Adding the uni-directional assumption

$$p_{12}^{*}=\frac{r_{12}+\left(\beta_2^3-\beta_1^2\right)+\left(\alpha_1^2+\alpha_1^3\right)\left(1+\beta_1^2-\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

    * Yields

$$LB_{12}^{TI,u} = \min\left\{\frac{r_{12}-\tilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\geq 0 \qquad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

$$UB_{12}^{TI,u} = \max\left\{\frac{r_{12}+\tilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\leq 1 \qquad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

    * Proof: Same as above except now $\ddot{Q}$ is not feasible.

- Under Temporal Invariance

$$p_{12}^{*}=\frac{r_{12}+\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)\left(\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)}{p_1+3\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)}$$

  – Yields

$$\begin{aligned}
LB_{12}^{TIV} &= \min\left\{\frac{r_{12}-\widetilde{Q}}{p_1},\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\right\}\geq 0 \\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2},\sqrt{1-r_{12}},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases} \\
UB_{12}^{TIV} &= \max\left\{\frac{r_{12}+\widetilde{Q}}{p_1},\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{12}},p_1/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

  – Proof:

    1. Evaluate $\partial LB_{12}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_1+3\widetilde{\widetilde{Q}}\right)-3\left(r_{12}+\widetilde{\widetilde{Q}}^2\right)\right) \\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{12}\right) \\
\Rightarrow sgn\left(\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{12}\right)<0 \\
\Rightarrow\widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

    2. Minimize LB_{12}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{12}=0 \\
\Rightarrow\ \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2}
\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

  – Adding the uni-directional assumption

$$p_{12}^{*}=\frac{r_{12}+\left(\theta_1^3+\theta_2^3\right)+\left(\theta_1^2+\theta_1^3\right)\left(\theta_1^2-\theta_2^3\right)}{p_1+3\left(\theta_1^2+\theta_1^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{12}^{TIV,u} &= \frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\geq 0 \\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2},\sqrt{1-r_{12}},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases} \\
UB_{12}^{TIV,u} &= \frac{r_{12}+\widetilde{Q}}{p_1}\leq 1 \qquad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above.

A.1.3 p^*_{13}

$$p^*_{13} = \frac{r_{13} + \overbrace{\left[\underset{\bullet}{\theta^{11}_{13}} + \underset{\bullet}{\theta^{12}_{13}} + \theta^{21}_{13} + \theta^{22}_{13} + \theta^{23}_{13} + \theta^{31}_{13} + \theta^{32}_{13} + \theta^{33}_{13}\right]}^{Q_{1,13}} - \overbrace{\left[\underset{\bullet}{\theta^{13}_{11}} + \underset{\bullet}{\theta^{13}_{12}} + \theta^{13}_{21} + \theta^{13}_{22} + \theta^{13}_{23} + \theta^{13}_{31} + \theta^{13}_{32} + \theta^{13}_{33}\right]}^{Q_{2,13}}}{p_1 + \underbrace{\left[\theta^{21}_{11} + \theta^{22}_{11} + \theta^{23}_{11} + \theta^{31}_{11} + \theta^{32}_{11} + \theta^{33}_{11} + \underset{\bullet}{\theta^{21}_{12}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{31}_{12}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{21}_{13}} + \underset{\bullet}{\theta^{22}_{13}} + \underset{\bullet}{\theta^{23}_{13}} + \underset{\bullet}{\theta^{31}_{13}} + \underset{\bullet}{\theta^{32}_{13}} + \underset{\bullet}{\theta^{33}_{13}}\right]}_{Q_{3,1}} - \underbrace{\left[\theta^{11}_{21} + \theta^{11}_{22} + \theta^{11}_{23} + \underset{\bullet}{\theta^{12}_{21}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{21}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{13}_{23}} + \theta^{11}_{31} + \theta^{11}_{32} + \theta^{11}_{33} + \underset{\bullet}{\theta^{12}_{31}} + \underset{\bullet}{\theta^{12}_{32}} + \underset{\bullet}{\theta^{12}_{33}} + \underset{\bullet}{\theta^{13}_{31}} + \underset{\bullet}{\theta^{13}_{32}} + \underset{\bullet}{\theta^{13}_{33}}\right]}_{Q_{4,1}}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{13} = \frac{r_{13} - \widetilde{Q}}{p_1} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{13} = \frac{r_{13} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{13} = \frac{r_{13} + \overbrace{\left[\theta^{23}_{13} + \theta^{33}_{13}\right]}^{Q^u_{1,12}} - \overbrace{\left[\underset{\bullet}{\theta^{13}_{11}} + \underset{\bullet}{\theta^{13}_{12}}\right]}^{Q^u_{2,12}}}{p_1 + \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{31}_{11}} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{33}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{12}} + \theta^{23}_{13} + \theta^{33}_{13}\right]}_{Q^u_{3,1}}}$$

- Yields

$$LB^u_{13} = \min\left\{\frac{r_{13} - \widetilde{Q}}{p_1}, \frac{r_{13}}{p_1 + \widetilde{\widetilde{Q}}}\right\} \geq 0 \qquad \widetilde{\widetilde{Q}} = \min\{1 - p_1, \widetilde{Q}\}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB^u_{13} = \frac{r_{13} + \widetilde{\widetilde{Q}}}{p_1 + \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\left\{1 - p_1, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{13}^{*}=\frac{r_{13}+\overbrace{\left[\alpha_1^1\beta_3^1+\alpha_1^1\beta_3^2+\alpha_1^2\beta_3^1+\alpha_1^2\beta_3^2+\alpha_1^2\beta_3^3+\alpha_1^3\beta_3^1+\alpha_1^3\beta_3^2+\alpha_1^3\beta_3^3\right]}^{Q_{1,13}}-\overbrace{\left[\alpha_1^1\beta_1^3+\alpha_1^1\beta_2^3+\alpha_2^1\beta_1^3+\alpha_2^1\beta_2^3+\alpha_2^1\beta_3^3+\alpha_3^1\beta_1^3+\alpha_3^1\beta_2^3+\alpha_3^1\beta_3^3\right]}^{Q_{2,13}}}{p_1+\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_1^3+\alpha_1^3\beta_1^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_1^3+\alpha_1^2\beta_2^1+\alpha_1^2\beta_2^2+\alpha_1^2\beta_2^3+\alpha_1^3\beta_2^1+\alpha_1^3\beta_2^2+\alpha_1^3\beta_2^3+\alpha_1^2\beta_3^1+\alpha_1^2\beta_3^2+\alpha_1^2\beta_3^3+\alpha_1^3\beta_3^1+\alpha_1^3\beta_3^2+\alpha_1^3\beta_3^3\right]}_{Q_{3,1}}-\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_2^1+\alpha_2^1\beta_3^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_2^2+\alpha_2^1\beta_3^2+\alpha_2^1\beta_1^3+\alpha_2^1\beta_2^3+\alpha_2^1\beta_3^3+\alpha_3^1\beta_1^1+\alpha_3^1\beta_2^1+\alpha_3^1\beta_3^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_2^2+\alpha_3^1\beta_3^2+\alpha_3^1\beta_1^3+\alpha_3^1\beta_2^3+\alpha_3^1\beta_3^3\right]}_{Q_{4,1}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,13} &= \left(\alpha_1^2+\alpha_1^3\right)+\left(\beta_3^1+\beta_3^2\right)\left(1-\alpha_1^2-\alpha_1^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_1^3\right)+\left(\theta_3^1+\theta_3^2\right)\left(1-\theta_1^2-\theta_1^3\right) && \text{(TIV)}\\
Q_{2,13} &= \left(\alpha_2^1+\alpha_3^1\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)+\left(\beta_1^3+\beta_2^3\right)\left(1-\alpha_1^2-\alpha_1^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_3^1\right)\left(1+\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^3+\theta_2^3\right)\left(1-\theta_1^2-\theta_1^3\right) && \text{(TIV)}\\
Q_{3,1} &= 3\left(\alpha_1^2+\alpha_1^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_1^3\right) && \text{(TIV)}\\
Q_{4,1} &= 3\left(\alpha_2^1+\alpha_3^1\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{13}^{*}=\frac{r_{13}+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

  - Yields

$$LB_{13}^{TI} = \min\left\{\frac{r_{13}-\ddot{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{13},p_1/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{13}^{TI} = \max\left\{\frac{r_{13}+\tilde{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{13},p_1/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{13}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_1^3,\beta_3^3=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_1+3\tilde{\tilde{Q}}\right)-3\left(r_{13}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_1-3r_{13}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{13}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{13}-p_1\right)\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{13}^{*}=\frac{r_{13}-\left(\beta_1^3+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right)\left(1+\beta_1^3+\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

    * Yields

$$LB_{13}^{TI,u} = \min\left\{\frac{r_{13}-\ddot{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{13}^{TI,u} = \frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\leq 1 \qquad \tilde{\tilde{Q}}=\begin{cases}0 & r_{13}\geq p_1/3\\ \min\left\{(1-p_1)/3,\tilde{Q}\right\} & \text{otherwise}\end{cases},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{13}^* = \frac{r_{13} + \left(\theta_3^2 + \theta_1^2 - \theta_2^1 - \theta_2^3\right) + \left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)\left(\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2\right)}{p_1 + 3\left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)}$$

  - Yields

$$\begin{aligned} LB_{13}^{TIV} &= \min\left\{\frac{r_{13} - \widetilde{Q}}{p_1}, \frac{r_{13} + \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\ & \qquad \widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2}, \sqrt{1 - r_{13}}, (1-p_1)/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\ UB_{13}^{TIV} &= \max\left\{\frac{r_{13} + \widetilde{Q}}{p_1}, \frac{r_{13} + \widetilde{\widetilde{Q}}^2}{p_1 - 3\widetilde{\widetilde{Q}}}\right\} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\left\{\sqrt{1 - r_{13}}, p_1/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} 2 - \sqrt{4 - Q} & \text{AE} \\ \left(3 - \sqrt{9 - 2Q}\right)/2 & \text{UE} \end{cases} \end{aligned}$$

  - Proof:

    1. Evaluate $\partial LB_{13}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_1 + 3\widetilde{\widetilde{Q}}\right) - 3\left(r_{13} + \widetilde{\widetilde{Q}}^2\right)\right) \\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_1 + \widetilde{\widetilde{Q}}\right) - r_{13}\right) \\ \Rightarrow sgn\left(\frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{13}\right) < 0 \\ \Rightarrow \widetilde{\widetilde{Q}} &> 0 \end{aligned}$$

    2. Minimize LB_{13}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned} \frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_1 + \widetilde{\widetilde{Q}}\right) - r_{13} = 0 \\ \Rightarrow \widetilde{\widetilde{Q}}^* &= \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2} \end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^*$. Thus, $\frac{r_{13} + \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^*$.

– Adding the uni-directional assumption

$$p^*_{13} = \frac{r_{13} + \left(\theta^2_1 - \theta^3_2\right) + \left(\theta^2_1 + \theta^3_1\right)\left(\theta^3_1 + \theta^3_2\right)}{p_1 + 3\left(\theta^2_1 + \theta^3_1\right)}$$

∗ Yields

$$LB^{TIV,u}_{13} = \min\left\{\frac{r_{13} - \widetilde{Q}}{p_1}, \frac{r_{13} + \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2}, \sqrt{1 - r_{13}}, (1 - p_1)/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB^{TIV,u}_{13} = \frac{r_{13} + \widetilde{\widetilde{Q}}}{p_1 + 3\widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{13} \geq p_1/3 \\ \min\left\{(1 - p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases},\ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

∗ Proof: Evaluate $\partial\left(\frac{r_{13}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{13}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_1 + 3\widetilde{\widetilde{Q}}\right) - 3\left(r_{13} + \widetilde{\widetilde{Q}}\right)\right) \\ &= sgn\left(p_1 - 3r_{13}\right) \end{aligned}$$

A.1.4 p^*_{21}

$$p^*_{21} = \frac{r_{21} + \overbrace{\left[\theta^{11}_{21} + \theta^{12}_{21} + \theta^{13}_{21} + \underset{\bullet}{\theta^{22}_{21}} + \underset{\bullet}{\theta^{23}_{21}} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21}\right]}^{Q_{1,21}} - \overbrace{\left[\theta^{21}_{11} + \theta^{21}_{12} + \theta^{21}_{13} + \underset{\bullet}{\theta^{21}_{22}} + \underset{\bullet}{\theta^{21}_{23}} + \theta^{21}_{31} + \theta^{21}_{32} + \theta^{21}_{33}\right]}^{Q_{2,21}}}{p_2 + \underbrace{\left[\theta^{11}_{21} + \theta^{12}_{21} + \theta^{13}_{21} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21} + \underset{\bullet}{\theta^{11}_{22}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{31}_{22}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{11}_{23}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{23}} + \underset{\bullet}{\theta^{31}_{23}} + \underset{\bullet}{\theta^{32}_{23}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q_{3,2}} - \underbrace{\left[\theta^{21}_{11} + \theta^{21}_{12} + \theta^{21}_{13} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{22}_{13}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}} + \theta^{21}_{31} + \theta^{21}_{32} + \theta^{21}_{33} + \underset{\bullet}{\theta^{22}_{31}} + \underset{\bullet}{\theta^{22}_{32}} + \underset{\bullet}{\theta^{22}_{33}} + \underset{\bullet}{\theta^{23}_{31}} + \underset{\bullet}{\theta^{23}_{32}} + \underset{\bullet}{\theta^{23}_{33}}\right]}_{Q_{4,2}}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{21} = \frac{r_{21} - \widetilde{Q}}{p_2} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{21} = \frac{r_{21} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{21} = \frac{r_{21} + \overbrace{\left[\underset{\bullet}{\theta^{22}_{21}} + \underset{\bullet}{\theta^{23}_{21}} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21}\right]}^{Q^u_{1,21}} - \overbrace{\left[\theta^{21}_{11}\right]}^{Q^u_{2,21}}}{p_2 + \underbrace{\left[\theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q^u_{3,2}} - \underbrace{\left[\theta^{21}_{11} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}}\right]}_{Q^u_{4,2}}}$$

- Yields

$$LB^u_{21} = \min\left\{\frac{r_{21} - \widetilde{\widetilde{Q}}}{p_2 - \widetilde{\widetilde{Q}}}, \frac{r_{21}}{p_2 + \widehat{Q}}\right\}, \geq 0 \qquad \widetilde{\widetilde{Q}} = \min\{r_{21}, p_2, \widetilde{Q}\},\ \widehat{Q} = \min\{1 - p_2, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{21} = \frac{r_{21} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, \widetilde{Q}\} & \text{UE} \end{cases},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{21}^{*}=\frac{r_{21}+\overbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^2\beta_1^2+\alpha_2^2\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3\right]}^{Q_{1,21}}-\overbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_2^2\beta_2^1+\alpha_2^2\beta_3^1+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1\right]}^{Q_{2,21}}}{\begin{array}{c}p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}}\\-\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,21} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_2^3\right)+\left(\theta_1^2+\theta_1^3\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{2,21} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)+\left(\beta_2^1+\beta_3^1\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_3^2\right)\left(1+\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_3^1\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{21}^{*}=\frac{r_{21}+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)+\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)}{p_2+3\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)}$$

  - Yields

$$\begin{aligned}LB_{21}^{TI} &= \min\left\{\frac{r_{21}-\ddot{Q}}{p_2},\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\ &\quad \tilde{\tilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\ UB_{21}^{TI} &= \max\left\{\frac{r_{21}+\tilde{Q}}{p_2},\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1\\ &\quad \tilde{\tilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{21}-\ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_3^2=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_2+3\tilde{\tilde{Q}}\right)-3\left(r_{21}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_2-3r_{21}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2-3\widehat{Q}\right)+3\left(r_{21}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{21}-p_2\right)\end{aligned}$$

– Adding the uni-directional assumption

$$p_{21}^{*}=\frac{r_{21}+\left(\beta_{1}^{2}+\beta_{1}^{3}\right)+\left(\alpha_{2}^{3}-\alpha_{1}^{2}\right)\left(1-\beta_{1}^{2}-\beta_{1}^{3}\right)}{p_{2}+3\left(\alpha_{2}^{3}-\alpha_{1}^{2}\right)}$$

∗ Yields

$$LB_{21}^{TI,u} = \min\left\{\frac{r_{21}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

$$UB_{21}^{TI,u} = \max\left\{\frac{r_{21}+\widetilde{Q}}{p_2},\frac{r_{21}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1$$

$$\widetilde{\widetilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}$$

∗ Proof: UB is same as above except now $\ddot{Q}$ is not feasible. The LB is not r_{21}/p_2 as the derivative of one of the terms in $\min\{\cdot\}$ wrt Q must be negative.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{21}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(p_2+3\widetilde{\widetilde{Q}}-3\left(r_{21}+\widetilde{\widetilde{Q}}\right)\right)\\ &= sgn\left(p_2-3r_{21}\right)\\ sgn\left(\frac{\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(p_2+3\widehat{Q}-3\left(r_{21}+\widehat{Q}\right)\right)\\ &= sgn\left(3r_{21}-p_2\right)\end{aligned}$$

- Under Temporal Invariance

$$p_{21}^{*} = \frac{r_{21} + \left(\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2\right) + \left(\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2\right)\left(\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3\right)}{p_2 + 3\left(\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2\right)}$$

  - Yields

$$LB_{21}^{TIV} = \min\left\{\frac{r_{21} - \widetilde{Q}}{p_2}, \frac{r_{21} + \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{21}}}{2}, \sqrt{1 - r_{21}}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{21}^{TIV} = \max\left\{\frac{r_{21} + \widetilde{Q}}{p_2}, \frac{r_{21} + \widetilde{\widetilde{Q}}^2}{p_2 - 3\widetilde{\widetilde{Q}}}\right\} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\left\{\sqrt{1 - r_{21}}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

  - Proof:

    1. Evaluate $\partial LB_{21}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_2 + 3\widetilde{\widetilde{Q}}\right) - 3\left(r_{21} + \widetilde{\widetilde{Q}}^2\right)\right) \\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_2 + \widetilde{\widetilde{Q}}\right) - r_{21}\right) \\ \Rightarrow sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{21}\right) < 0 \\ \Rightarrow \widetilde{\widetilde{Q}} &> 0 \end{aligned}$$

    2. Minimize LB_{21}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned} \frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_2 + \widetilde{\widetilde{Q}}\right) - r_{21} = 0 \\ \Rightarrow \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{21}}}{2} \end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{21} + \widetilde{\widetilde{Q}}^2}{p_2 + 3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

– Adding the uni-directional assumption

$$p^*_{21} = \frac{r_{21} + \left(\theta_1^3 + \theta_2^3\right) - \left(\theta_2^3 - \theta_1^2\right)\left(\theta_1^2 + \theta_1^3\right)}{p_2 + 3\left(\theta_2^3 - \theta_1^2\right)}$$

* Yields

$$LB_{21}^{TIV,u} = \frac{r_{21} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}} \geq 0$$

$$\widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{21} < p_2/3 \\ \min\left\{1 - r_{21}, (1-p_2)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \quad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{21}^{TIV,u} = \max\left\{\frac{r_{21} + \widetilde{Q}}{p_2}, \frac{r_{21} + \widetilde{\widetilde{Q}}^2}{p_2 - 3\widetilde{\widetilde{Q}}}\right\} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\left\{\sqrt{1 - r_{21}}, p_2/3, \widetilde{Q}\right\}, \quad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

* Proof: Evaluate $\partial LB_{21}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(p_2 + 3\widetilde{\widetilde{Q}} - 3\left(r_{21} + \widetilde{\widetilde{Q}}\right)\right) \\ &= sgn\left(p_2 - 3r_{21}\right) \end{aligned}$$

A.1.5 p_{22}^*

$$p_{22}^* = \frac{r_{22} + \overbrace{\left[\theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \underset{\bullet}{\theta_{22}^{21}} + \underset{\bullet}{\theta_{22}^{23}} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33}\right]}^{Q_{1,22}} - \overbrace{\left[\theta_{11}^{22} + \theta_{12}^{22} + \theta_{13}^{22} + \underset{\bullet}{\theta_{21}^{22}} + \underset{\bullet}{\theta_{23}^{22}} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22}\right]}^{Q_{2,22}}}{\begin{array}{l} p_2 + \underbrace{\left[\underset{\bullet}{\theta_{21}^{11}} + \underset{\bullet}{\theta_{21}^{12}} + \underset{\bullet}{\theta_{21}^{13}} + \underset{\bullet}{\theta_{21}^{31}} + \underset{\bullet}{\theta_{21}^{32}} + \underset{\bullet}{\theta_{21}^{33}} + \theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33} + \underset{\bullet}{\theta_{23}^{11}} + \underset{\bullet}{\theta_{23}^{12}} + \underset{\bullet}{\theta_{23}^{13}} + \underset{\bullet}{\theta_{23}^{31}} + \underset{\bullet}{\theta_{23}^{32}} + \underset{\bullet}{\theta_{23}^{33}}\right]}_{Q_{3,2}} \\ \quad - \underbrace{\left[\underset{\bullet}{\theta_{11}^{21}} + \underset{\bullet}{\theta_{12}^{21}} + \underset{\bullet}{\theta_{13}^{21}} + \theta_{11}^{22} + \theta_{12}^{22} + \theta_{13}^{22} + \underset{\bullet}{\theta_{11}^{23}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{31}^{21}} + \underset{\bullet}{\theta_{32}^{21}} + \underset{\bullet}{\theta_{33}^{21}} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22} + \underset{\bullet}{\theta_{31}^{23}} + \underset{\bullet}{\theta_{32}^{23}} + \underset{\bullet}{\theta_{33}^{23}}\right]}_{Q_{4,2}} \end{array}}$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{22} = \frac{r_{22} - \widetilde{Q}}{p_2} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{22} = \frac{r_{22} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p_{22}^* = \frac{r_{22} + \overbrace{\left[\underset{\bullet}{\theta_{22}^{23}} + \theta_{22}^{32} + \theta_{22}^{33}\right]}^{Q_{1,22}^u} - \overbrace{\left[\theta_{11}^{22} + \theta_{12}^{22} + \underset{\bullet}{\theta_{21}^{22}}\right]}^{Q_{2,22}^u}}{p_2 + \underbrace{\left[\underset{\bullet}{\theta_{21}^{31}} + \underset{\bullet}{\theta_{21}^{32}} + \underset{\bullet}{\theta_{21}^{33}} + \theta_{22}^{32} + \theta_{22}^{33} + \underset{\bullet}{\theta_{23}^{33}}\right]}_{Q_{3,2}^u} - \underbrace{\left[\underset{\bullet}{\theta_{11}^{21}} + \theta_{11}^{22} + \theta_{12}^{22} + \underset{\bullet}{\theta_{11}^{23}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{13}^{23}}\right]}_{Q_{4,2}^u}}$$

- Yields

$$LB_{22}^u = \frac{r_{22} - \widetilde{Q}}{p_2} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{22}^u = \frac{r_{22} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{22}^{*}=\frac{r_{22}+\overbrace{\left[\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^2\beta_2^1+\alpha_2^2\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3\right]}^{Q_{1,22}}-\overbrace{\left[\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_2^2\beta_1^2+\alpha_2^2\beta_3^2+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2\right]}^{Q_{2,22}}}{\begin{array}{c}p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}}\\-\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,22} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_2^1+\beta_2^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= 2\left(\theta_2^1+\theta_2^3\right)-\left(\theta_2^1+\theta_2^3\right)^2 && \text{(TIV)}\\
Q_{2,22} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)+\left(\beta_1^2+\beta_3^2\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= 2\left(\theta_1^2+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right)+\left(\theta_1^2+\theta_3^2\right)^2 && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{22}^{*} = \frac{r_{22} + \left(\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2\right) + \left(\beta_2^1 + \beta_2^3 - \beta_1^2 - \beta_3^2\right)\left(1 + \alpha_1^2 + \alpha_3^2 - \alpha_2^1 - \alpha_2^3\right)}{p_2 + 3\left(\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2\right)}$$

  - Yields

$$\begin{aligned} LB_{22}^{TI} &= \min\left\{\frac{r_{22} - \ddot{Q}}{p_2}, \frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \geq 0 \\ & \tilde{\tilde{Q}} = \min\left\{1 - r_{22}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{22}, p_2/3, \ddot{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases},\ \ddot{Q} = \begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE} \end{cases} \\ UB_{22}^{TI} &= \max\left\{\frac{r_{22} + \widetilde{Q}}{p_2}, \frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \leq 1 \\ & \tilde{\tilde{Q}} = \min\left\{1 - r_{22}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{22}, p_2/3, \ddot{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases},\ \ddot{Q} = \begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE} \end{cases} \end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{22} - \ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_1^2, \beta_3^2 = Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2, \alpha_3^2 = Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_2 + 3\tilde{\tilde{Q}}\right) - 3\left(r_{22} + \tilde{\tilde{Q}}\right)\right) \\ &= sgn\left(p_2 - 3r_{22}\right) \end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2 - 3\widehat{Q}\right) + 3\left(r_{22} - \widehat{Q}\right)\right) \\ &= sgn\left(3r_{22} - p_2\right) \end{aligned}$$

  - Adding the uni-directional assumption

$$p_{22}^{*} = \frac{r_{22} + \left(\alpha_2^3 - \alpha_1^2\right) + \left(\beta_2^3 - \beta_1^2\right)\left(1 + \alpha_1^2 - \alpha_2^3\right)}{p_2 + 3\left(\alpha_2^3 - \alpha_1^2\right)}$$

    * Yields

$$\begin{aligned} LB_{22}^{TI,u} &= \min\left\{\frac{r_{22} - \widetilde{Q}}{p_2}, \frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \geq 0 \\ & \tilde{\tilde{Q}} = \min\left\{1 - r_{22}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{22}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases} \\ UB_{22}^{TI,u} &= \max\left\{\frac{r_{22} + \widetilde{Q}}{p_2}, \frac{r_{22} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{22} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \leq 1 \\ & \tilde{\tilde{Q}} = \min\left\{1 - r_{22}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{22}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases} \end{aligned}$$

    * Proof: Same as above except now $\ddot{Q}$ is not feasible.

- Under Temporal Invariance

$$p_{22}^{*}=\frac{r_{22}+2\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)-\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)^2}{p_2+3\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)}$$

  - Yields

$$\begin{aligned}
LB_{22}^{TIV} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\min\left\{(1-p_2)/3,\widetilde{Q}\right\},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}\geq 2p_2/3\\ \min\left\{\frac{(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(-1+\sqrt{1+r_{22}}),p_2/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{22}^{TIV} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\begin{cases}0 & r_{22}\geq 2p_2/3\\ \min\left\{\frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(1-p_2)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}<2p_2/3\\ \min\left\{\frac{(2/3)p_2-\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(-1+\sqrt{1+r_{22}}),p_2/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(-2-2\tilde{\tilde{Q}}\right)\left(p_2-3\tilde{\tilde{Q}}\right)+3\left(r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\right)\right) \\
&= sgn\left(-(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=0} &= sgn\left(-(2/3)p_2+r_{22}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=1} &= sgn\left(-(4/3)p_2+1+r_{22}\right) \gtrless 0
\end{aligned}$$

2. Ensure $r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\geq 0$

$$\begin{aligned}
& r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\geq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}}^2+2\tilde{\tilde{Q}}-r_{22}\leq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}}\leq\frac{-2+\sqrt{4+4r_{22}}}{2} \\
\Rightarrow\ & \tilde{\tilde{Q}}\leq -1+\sqrt{1+r_{22}}
\end{aligned}$$

3. Minimize $\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{22}<2p_2/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}=0 \\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{22}\geq 2p_2/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}=0 \\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_2-\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}=.$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$
\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_2+3\widehat{Q}\right)-3\left(r_{22}+2\widehat{Q}-\widehat{Q}^2\right)\right) \\
&= sgn\left((2/3)p_2\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{22}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_2-r_{22}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{22}\right) < 0
\end{aligned}
$$

6. Maximize $\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{22} < 2p_2/3$

$$
\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\tilde{\tilde{Q}}}\right) &\propto (2/3)p_2\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{22}=0 \\
\Rightarrow \tilde{\tilde{Q}}^* &= \frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}
$$

7. Minimize $\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}} \Rightarrow \widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{221}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}} < \frac{r_{22}}{p_2}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p^*_{22} = \frac{r_{22} + 2\left(\theta^3_2 - \theta^2_1\right) - \left(\theta^3_2 - \theta^2_1\right)^2}{p_2 + 3\left(\theta^3_2 - \theta^2_1\right)}$$

* Yields

$$\begin{aligned}
LB^{TIV,u}_{22} &= \min\left\{\frac{r_{22} + 2\widehat{Q} - \widehat{Q}^2}{p_2 + 3\widehat{Q}}, \frac{r_{22} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_2 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \qquad \widehat{Q} = \min\left\{(1-p_2)/3, \widetilde{Q}\right\}, \\
& \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{22} \geq 2p_2/3 \\ \min\left\{\frac{(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}, (-1 + \sqrt{1 + r_{22}}), p_2/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\
UB^{TIV,u}_{22} &= \min\left\{\frac{r_{22} + 2\widehat{Q} - \widehat{Q}^2}{p_2 + 3\widehat{Q}}, \frac{r_{22} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_2 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \qquad \widehat{Q} = \begin{cases} 0 & r_{22} \geq 2p_2/3 \\ \min\left\{\frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}, (1 - p_2)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{22} < 2p_2/3 \\ \min\left\{(-1 + \sqrt{1 + r_{22}}), p_2/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}
\end{aligned}$$

* Proof: Same as above.

A.1.6 p_{23}^*

$$p_{23}^* = \frac{r_{23} + \overbrace{\left[\theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \underset{\bullet}{\theta_{23}^{21}} + \underset{\bullet}{\theta_{23}^{22}} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33}\right]}^{Q_{1,23}} - \overbrace{\left[\theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \underset{\bullet}{\theta_{21}^{23}} + \underset{\bullet}{\theta_{22}^{23}} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23}\right]}^{Q_{2,23}}}{\begin{array}{l} p_2 + \underbrace{\left[\underset{\bullet}{\theta_{21}^{11}} + \underset{\bullet}{\theta_{21}^{12}} + \underset{\bullet}{\theta_{21}^{13}} + \underset{\bullet}{\theta_{21}^{31}} + \underset{\bullet}{\theta_{21}^{32}} + \underset{\bullet}{\theta_{21}^{33}} + \underset{\bullet}{\theta_{22}^{11}} + \underset{\bullet}{\theta_{22}^{12}} + \underset{\bullet}{\theta_{22}^{13}} + \underset{\bullet}{\theta_{22}^{31}} + \underset{\bullet}{\theta_{22}^{32}} + \underset{\bullet}{\theta_{22}^{33}} + \theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33}\right]}_{Q_{3,2}} \\ \quad - \underbrace{\left[\underset{\bullet}{\theta_{11}^{21}} + \underset{\bullet}{\theta_{12}^{21}} + \underset{\bullet}{\theta_{13}^{21}} + \underset{\bullet}{\theta_{11}^{22}} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{13}^{22}} + \theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \underset{\bullet}{\theta_{31}^{21}} + \underset{\bullet}{\theta_{32}^{21}} + \underset{\bullet}{\theta_{33}^{21}} + \underset{\bullet}{\theta_{31}^{22}} + \underset{\bullet}{\theta_{32}^{22}} + \underset{\bullet}{\theta_{33}^{22}} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23}\right]}_{Q_{4,2}} \end{array}}$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{23} = \frac{r_{23} - \widetilde{Q}}{p_2} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{23} = \frac{r_{23} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p_{23}^* = \frac{r_{23} + \overbrace{\left[\theta_{23}^{33}\right]}^{Q_{1,23}^u} - \overbrace{\left[\theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \underset{\bullet}{\theta_{21}^{23}} + \underset{\bullet}{\theta_{22}^{23}}\right]}^{Q_{2,23}^u}}{p_2 + \underbrace{\left[\underset{\bullet}{\theta_{21}^{31}} + \underset{\bullet}{\theta_{21}^{32}} + \underset{\bullet}{\theta_{21}^{33}} + \underset{\bullet}{\theta_{22}^{32}} + \underset{\bullet}{\theta_{22}^{33}} + \theta_{23}^{33}\right]}_{Q_{3,2}^u} - \underbrace{\left[\underset{\bullet}{\theta_{11}^{21}} + \underset{\bullet}{\theta_{11}^{22}} + \underset{\bullet}{\theta_{12}^{22}} + \theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23}\right]}_{Q_{4,2}^u}}$$

- Yields

$$LB_{23}^u = \frac{r_{23} - \widetilde{Q}}{p_2} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{23}^u = \max\left\{\frac{r_{23}}{p_2 - \widehat{Q}}, \frac{r_{23} + \widetilde{\widetilde{Q}}}{p_2 + \widetilde{\widetilde{Q}}}\right\} \leq 1 \qquad \widehat{Q} = \min\left\{p_2, \widetilde{Q}\right\}, \ \widetilde{\widetilde{Q}} = \min\left\{1 - p_2, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{23}^{*}=\frac{r_{23}+\overbrace{\left[\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^2\beta_3^1+\alpha_2^2\beta_3^2+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}^{Q_{1,23}}-\overbrace{\left[\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_2^2\beta_1^3+\alpha_2^2\beta_2^3+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}^{Q_{2,23}}}{\begin{array}{c}p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}} \\ -\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,23} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_3^1+\beta_3^2\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_2^3\right)+\left(\theta_3^1+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{2,23} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)+\left(\beta_1^3+\beta_2^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_3^2\right)\left(1+\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^3+\theta_2^3\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{23}^{*}=\frac{r_{23}+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)}{p_2+3\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)}$$

  - Yields

$$LB_{23}^{TI} = \min\left\{\frac{r_{23}-\ddot{Q}}{p_1},\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{23}^{TI} = \max\left\{\frac{r_{23}+\widetilde{Q}}{p_2},\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{23}-\ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_2^3,\beta_1^3=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_3^2=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_2+3\tilde{\tilde{Q}}\right)-3\left(r_{23}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_2-3r_{23}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2-3\widehat{Q}\right)+3\left(r_{23}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{23}-p_2\right)\end{aligned}$$

– Adding the uni-directional assumption

$$p_{23}^* = \frac{r_{23} - \left(\beta_1^3 + \beta_2^3\right) + \left(\alpha_2^3 - \alpha_1^2\right)\left(1 + \beta_1^3 + \beta_2^3\right)}{p_2 + 3\left(\alpha_2^3 - \alpha_1^2\right)}$$

* Yields

$$LB_{23}^{TI,u} = \min\left\{\frac{r_{23} - \ddot{Q}}{p_2}, \frac{r_{23} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}}, \frac{r_{23} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{1 - r_{23}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{23}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases},\ \ddot{Q} = \begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE} \end{cases}$$

$$UB_{23}^{TI,u} = \max\left\{\frac{r_{23} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}}, \frac{r_{23} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\}$$

$$\widetilde{\widetilde{Q}} = \min\left\{1 - r_{23}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{23}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE} \end{cases}$$

* Proof: LB is the same except that $\ddot{Q}$ is no longer feasible in $\widehat{Q}$. The UB is not r_{23}/p_2 as the derivative of one of the terms in $\max\{\cdot\}$ wrt Q must be positive.

$$\begin{aligned} sgn\left(\frac{\partial \frac{r_{23} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}}}{\partial \widetilde{\widetilde{Q}}}\right) &= sgn\left(p_2 + 3\widetilde{\widetilde{Q}} - 3\left(r_{23} + \widetilde{\widetilde{Q}}\right)\right) \\ &= sgn\left(p_2 - 3r_{23}\right) \\ sgn\left(\frac{\partial \frac{r_{23} - \widehat{Q}}{p_2 - 3\widehat{Q}}}{\partial \widehat{Q}}\right) &= sgn\left(p_2 + 3\widehat{Q} - 3\left(r_{23} + \widehat{Q}\right)\right) \\ &= sgn\left(3r_{23} - p_2\right) \end{aligned}$$

- Under Temporal Invariance

$$p_{23}^{*}=\frac{r_{23}+\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)}{p_2+3\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)}$$

  - Yields

$$LB_{23}^{TIV} = \min\left\{\frac{r_{23}-\widetilde{Q}}{p_2},\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{23}}}{2},\sqrt{1-r_{23}},(1-p_2)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{23}^{TIV} = \max\left\{\frac{r_{23}+\widetilde{Q}}{p_2},\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{23}},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  - Proof:

    1. Evaluate $\partial LB_{23}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_2+3\widetilde{\widetilde{Q}}\right)-3\left(r_{23}+\widetilde{\widetilde{Q}}^2\right)\right)\\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{23}\right)\\
\Rightarrow sgn\left(\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{23}\right)<0\\
\Rightarrow\widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

    2. Minimize LB_{23}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{23}=0\\
\Rightarrow\quad \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{23}}}{2}
\end{aligned}$$

So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

– Adding the uni-directional assumption

$$p_{23}^{*}=\frac{r_{23}-\left(\theta_{1}^{2}+\theta_{1}^{3}\right)+\left(\theta_{2}^{3}-\theta_{1}^{2}\right)\left(\theta_{1}^{3}+\theta_{2}^{3}\right)}{p_{2}+3\left(\theta_{2}^{3}-\theta_{1}^{2}\right)}$$

* Yields

$$LB_{23}^{TIV,u} = \min\left\{\frac{r_{23}-\widetilde{Q}}{p_{2}},\frac{r_{23}+\widetilde{\widetilde{Q}}^{2}}{p_{2}+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_{2}+\sqrt{(4/9)p_{2}^{2}+4r_{23}}}{2},\sqrt{1-r_{23}},(1-p_{2})/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{23}^{TIV,u} = \frac{r_{23}-\widetilde{\widetilde{Q}}}{p_{2}-3\widetilde{\widetilde{Q}}}\leq 1$$

$$\widetilde{\widetilde{Q}}=\begin{cases}0 & r_{23}<p_{2}/3\\ \min\left\{r_{23},p_{2}/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

* Proof: Evaluate $\partial\left(\frac{r_{23}-\widetilde{\widetilde{Q}}}{p_{2}-3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{23}-\widetilde{\widetilde{Q}}}{p_{2}-3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(-\left(p_{2}-3\widetilde{\widetilde{Q}}\right)+3\left(r_{23}-\widetilde{\widetilde{Q}}\right)\right)\\ &= sgn\left(3r_{23}-p_{2}\right)\end{aligned}$$

A.1.7 p^*_{31}

$$p^*_{31} = \frac{r_{31} + \overbrace{\left[\theta^{11}_{31} + \theta^{12}_{31} + \theta^{13}_{31} + \theta^{21}_{31} + \theta^{22}_{31} + \theta^{23}_{31} + \underset{\bullet}{\theta^{32}_{31}} + \underset{\bullet}{\theta^{33}_{31}}\right]}^{Q_{1,31}} - \overbrace{\left[\theta^{31}_{11} + \theta^{31}_{12} + \theta^{31}_{13} + \theta^{31}_{21} + \theta^{31}_{22} + \theta^{31}_{23} + \underset{\bullet}{\theta^{31}_{32}} + \underset{\bullet}{\theta^{31}_{33}}\right]}^{Q_{2,31}}}{\begin{array}{l} p_3 + \underbrace{\left[\theta^{11}_{31} + \theta^{12}_{31} + \theta^{13}_{31} + \theta^{21}_{31} + \theta^{22}_{31} + \theta^{23}_{31} + \underset{\bullet}{\theta^{11}_{32}} + \underset{\bullet}{\theta^{12}_{32}} + \underset{\bullet}{\theta^{13}_{32}} + \underset{\bullet}{\theta^{21}_{32}} + \underset{\bullet}{\theta^{22}_{32}} + \underset{\bullet}{\theta^{23}_{32}} + \underset{\bullet}{\theta^{11}_{33}} + \underset{\bullet}{\theta^{12}_{33}} + \underset{\bullet}{\theta^{13}_{33}} + \underset{\bullet}{\theta^{21}_{33}} + \underset{\bullet}{\theta^{22}_{33}} + \underset{\bullet}{\theta^{23}_{33}}\right]}_{Q_{3,3}} \\ - \underbrace{\left[\theta^{31}_{11} + \theta^{31}_{12} + \theta^{31}_{13} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{32}_{13}} + \underset{\bullet}{\theta^{33}_{11}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{33}_{13}} + \theta^{31}_{21} + \theta^{31}_{22} + \theta^{31}_{23} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{32}_{23}} + \underset{\bullet}{\theta^{33}_{21}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q_{4,3}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{31} &= \frac{r_{31} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{31} &= \frac{r_{31} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{31} = \frac{r_{31} + \overbrace{\left[\underset{\bullet}{\theta^{32}_{31}} + \underset{\bullet}{\theta^{33}_{31}}\right]}^{Q^u_{1,31}} - \overbrace{\left[\theta^{31}_{11} + \theta^{31}_{21}\right]}^{Q^u_{2,31}}}{p_3 - \underbrace{\left[\theta^{31}_{11} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{11}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{33}_{13}} + \theta^{31}_{21} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{21}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q^u_{4,3}}}$$

- Yields

$$\begin{aligned} LB^u_{31} &= \frac{r_{31} - \widetilde{\widetilde{Q}}}{p_3 - \widetilde{\widetilde{Q}}} \geq 0 & \qquad \widetilde{\widetilde{Q}} &= \min\left\{r_{31}, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{31} &= \frac{r_{31} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{31}^{*}=\frac{r_{31}+\overbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^3\beta_1^2+\alpha_3^3\beta_1^3\right]}^{Q_{1,31}}-\overbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_3^3\beta_2^1+\alpha_3^3\beta_3^1\right]}^{Q_{2,31}}}{p_3+\underbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^1\beta_2^1+\alpha_3^1\beta_2^2+\alpha_3^1\beta_2^3+\alpha_3^2\beta_2^1+\alpha_3^2\beta_2^2+\alpha_3^2\beta_2^3+\alpha_3^1\beta_3^1+\alpha_3^1\beta_3^2+\alpha_3^1\beta_3^3+\alpha_3^2\beta_3^1+\alpha_3^2\beta_3^2+\alpha_3^2\beta_3^3\right]}_{Q_{3,3}}-\underbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_2^2+\alpha_1^3\beta_3^2+\alpha_1^3\beta_1^3+\alpha_1^3\beta_2^3+\alpha_1^3\beta_3^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_2^2+\alpha_2^3\beta_3^2+\alpha_2^3\beta_1^3+\alpha_2^3\beta_2^3+\alpha_2^3\beta_3^3\right]}_{Q_{4,3}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,31} &= \left(\alpha_3^1+\alpha_3^2\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_3^1+\theta_3^2\right)+\left(\theta_1^2+\theta_1^3\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{2,31} &= \left(\alpha_1^3+\alpha_2^3\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)+\left(\beta_2^1+\beta_3^1\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_1^3+\theta_2^3\right)\left(1+\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_3^1\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{3,3} &= 3\left(\alpha_3^1+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_3^1+\theta_3^2\right) && \text{(TIV)}\\
Q_{4,3} &= 3\left(\alpha_1^3+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^3+\theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{31}^{*}=\frac{r_{31}+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)+\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)}{p_3+3\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)}$$

  – Yields

$$\begin{aligned}LB_{31}^{TI} &= \min\left\{\frac{r_{31}-\ddot{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\geq 0\\ &\quad \tilde{\tilde{Q}}=\min\left\{1-r_{31},(1-p_3)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\ UB_{31}^{TI} &= \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\leq 1\\ &\quad \tilde{\tilde{Q}}=\min\left\{1-r_{31},(1-p_3)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\end{aligned}$$

  – Proof:

    1. $\ddot{Q}$ in $\frac{r_{31}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^3,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_3+3\tilde{\tilde{Q}}\right)-3\left(r_{31}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_3-3r_{31}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}sgn\left(\frac{\partial\left(\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_3-3\widehat{Q}\right)+3\left(r_{31}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{31}-p_3\right)\end{aligned}$$

  – Adding the uni-directional assumption

$$p_{31}^{*}=\frac{r_{31}+\left(\beta_1^2+\beta_1^3\right)-\left(\alpha_1^3+\alpha_2^3\right)\left(1-\beta_1^2-\beta_1^3\right)}{p_3-3\left(\alpha_1^3+\alpha_2^3\right)}$$

    * Yields

$$\begin{aligned}LB_{31}^{TI} &= \frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\geq 0 \qquad \tilde{\tilde{Q}}=\begin{cases}0 & r_{31}\geq p_3/3\\ \min\left\{r_{31},p_3/3,\ddot{Q}\right\} & \text{otherwise}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\ UB_{31}^{TI} &= \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\right\}\leq 1\\ &\quad \tilde{\tilde{Q}}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{31}^{*}=\frac{r_{31}+\left(\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)+\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)}{p_3+3\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)}$$

  - Yields

$$LB_{31}^{TIV} = \min\left\{\frac{r_{31}-\widetilde{Q}}{p_3},\frac{r_{31}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{31}}}{2},\sqrt{1-r_{31}},(1-p_3)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{31}^{TIV} = \max\left\{\frac{r_{31}+\widetilde{Q}}{p_3},\frac{r_{31}+\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{31}},p_3/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  - Proof:

    1. Evaluate $\partial LB_{31}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_3+3\widetilde{\widetilde{Q}}\right)-3\left(r_{31}+\widetilde{\widetilde{Q}}^2\right)\right)\\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{31}\right)\\
\Rightarrow sgn\left(\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{31}\right)<0\\
\Rightarrow \widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

    2. Minimize LB_{31}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{31}=0\\
\Rightarrow &\quad \widetilde{\widetilde{Q}}^*=\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{31}}}{2}
\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^*$. Thus, $\frac{r_{31}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^*$.

- Adding the uni-directional assumption

$$p_{31}^{*}=\frac{r_{31}+\left(\theta_1^2-\theta_2^3\right)+\left(\theta_1^3+\theta_2^3\right)\left(\theta_1^2+\theta_1^3\right)}{p_3-3\left(\theta_1^3+\theta_2^3\right)}$$

  * Yields

$$LB_{31}^{TIV,u} = \frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}} \geq 0$$

$$\tilde{\tilde{Q}}=\begin{cases} 0 & r_{31}\geq p_3/3 \\ \min\left\{r_{31},p_3/3,\tilde{Q}\right\} & \text{otherwise}\end{cases}, \quad \tilde{Q}=\begin{cases} 3-\sqrt{9-Q} & \text{AE} \\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{31}^{TIV,u} = \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right\}\leq 1 \qquad \tilde{\tilde{Q}}=\min\left\{\sqrt{1-r_{31}},p_3/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases} 3-\sqrt{9-Q} & \text{AE} \\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  * Proof: Evaluate $\partial\left(LB_{31}^{TIV,u}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{31}^{TIV,u}}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(-\left(p_3-3\tilde{\tilde{Q}}\right)+3\left(r_{31}-\tilde{\tilde{Q}}\right)\right) \\ &= sgn\left(3r_{31}-p_3\right)\end{aligned}$$

A.1.8 p^*_{32}

$$p^*_{32} = \frac{r_{32} + \overbrace{\left[\theta^{11}_{32} + \theta^{12}_{32} + \theta^{13}_{32} + \theta^{21}_{32} + \theta^{22}_{32} + \theta^{23}_{32} + \underset{\bullet}{\theta^{31}_{32}} + \underset{\bullet}{\theta^{33}_{32}}\right]}^{Q_{1,32}} - \overbrace{\left[\theta^{32}_{11} + \theta^{32}_{12} + \theta^{32}_{13} + \theta^{32}_{21} + \theta^{32}_{22} + \theta^{32}_{23} + \underset{\bullet}{\theta^{32}_{31}} + \underset{\bullet}{\theta^{32}_{33}}\right]}^{Q_{2,32}}}{\begin{array}{l} p_3 + \underbrace{\left[\underset{\bullet}{\theta^{11}_{31}} + \underset{\bullet}{\theta^{12}_{31}} + \underset{\bullet}{\theta^{13}_{31}} + \underset{\bullet}{\theta^{21}_{31}} + \underset{\bullet}{\theta^{22}_{31}} + \underset{\bullet}{\theta^{23}_{31}} + \underset{\bullet}{\theta^{11}_{32}} + \underset{\bullet}{\theta^{12}_{32}} + \underset{\bullet}{\theta^{13}_{32}} + \underset{\bullet}{\theta^{21}_{32}} + \underset{\bullet}{\theta^{22}_{32}} + \underset{\bullet}{\theta^{23}_{32}} + \theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33}\right]}_{Q_{3,3}} \\ \quad - \underbrace{\left[\underset{\bullet}{\theta^{31}_{11}} + \underset{\bullet}{\theta^{31}_{12}} + \underset{\bullet}{\theta^{31}_{13}} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{32}_{13}} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \underset{\bullet}{\theta^{31}_{21}} + \underset{\bullet}{\theta^{31}_{22}} + \underset{\bullet}{\theta^{31}_{23}} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{32}_{23}} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q_{4,3}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{32} &= \frac{r_{32} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{32} &= \frac{r_{32} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{32} = \frac{r_{32} + \overbrace{\left[\underset{\bullet}{\theta^{33}_{32}}\right]}^{Q^u_{1,32}} - \overbrace{\left[\theta^{32}_{11} + \theta^{32}_{12} + \theta^{32}_{21} + \theta^{32}_{22} + \underset{\bullet}{\theta^{32}_{31}}\right]}^{Q^u_{2,32}}}{p_3 - \underbrace{\left[\theta^{31}_{11} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{11}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{33}_{13}} + \theta^{31}_{21} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{21}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q^u_{4,3}}}$$

- Yields

$$\begin{aligned} LB^u_{32} &= \frac{r_{32} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB^u_{32} &= \frac{r_{32} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases}, \ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{32}^{*} = \frac{r_{32} + \overbrace{\left[\alpha_3^1\beta_2^1 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_2^3 + \alpha_3^2\beta_2^1 + \alpha_3^2\beta_2^2 + \alpha_3^2\beta_2^3 + \alpha_3^3\beta_2^1 + \alpha_3^3\beta_2^3\right]}^{Q_{1,32}} - \overbrace{\left[\alpha_1^3\beta_1^2 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_3^2 + \alpha_2^3\beta_1^2 + \alpha_2^3\beta_2^2 + \alpha_2^3\beta_3^2 + \alpha_3^3\beta_1^2 + \alpha_3^3\beta_3^2\right]}^{Q_{2,32}}}{p_3 + \underbrace{\left[\alpha_3^1\beta_1^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_1^3 + \alpha_3^2\beta_1^1 + \alpha_3^2\beta_1^2 + \alpha_3^2\beta_1^3 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_2^3 + \alpha_3^2\beta_2^1 + \alpha_3^2\beta_2^2 + \alpha_3^2\beta_2^3 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_3^3 + \alpha_3^2\beta_3^1 + \alpha_3^2\beta_3^2 + \alpha_3^2\beta_3^3\right]}_{Q_{3,3}} - \underbrace{\left[\alpha_1^3\beta_1^1 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_1^3 + \alpha_1^3\beta_2^3 + \alpha_1^3\beta_3^3 + \alpha_2^3\beta_1^1 + \alpha_2^3\beta_2^1 + \alpha_2^3\beta_3^1 + \alpha_2^3\beta_1^2 + \alpha_2^3\beta_2^2 + \alpha_2^3\beta_3^2 + \alpha_2^3\beta_1^3 + \alpha_2^3\beta_2^3 + \alpha_2^3\beta_3^3\right]}_{Q_{4,3}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,32} &= \left(\alpha_3^1 + \alpha_3^2\right) + \left(\beta_2^1 + \beta_2^3\right)\left(1 - \alpha_3^1 - \alpha_3^2\right) && \text{(TI)} \\
&= \left(\theta_3^1 + \theta_3^2\right) + \left(\theta_2^2 + \theta_2^3\right)\left(1 - \theta_3^1 - \theta_3^2\right) && \text{(TIV)} \\
Q_{2,32} &= \left(\alpha_1^3 + \alpha_2^3\right)\left(1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3\right) + \left(\beta_1^2 + \beta_3^2\right)\left(1 - \alpha_3^1 - \alpha_3^2\right) && \text{(TI)} \\
&= \left(\theta_1^3 + \theta_2^3\right)\left(1 + \theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3\right) + \left(\theta_1^2 + \theta_3^2\right)\left(1 - \theta_3^1 - \theta_3^2\right) && \text{(TIV)} \\
Q_{3,3} &= 3\left(\alpha_3^1 + \alpha_3^2\right) && \text{(TI)} \\
&= 3\left(\theta_3^1 + \theta_3^2\right) && \text{(TIV)} \\
Q_{4,3} &= 3\left(\alpha_1^3 + \alpha_2^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^3 + \theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{32}^{*}=\frac{r_{32}+\left(\beta_{2}^{1}+\beta_{2}^{3}-\beta_{1}^{2}-\beta_{3}^{2}\right)+\left(\alpha_{3}^{1}+\alpha_{3}^{2}-\alpha_{1}^{3}-\alpha_{2}^{3}\right)\left(1+\beta_{1}^{2}+\beta_{3}^{2}-\beta_{2}^{1}-\beta_{2}^{3}\right)}{p_{3}+3\left(\alpha_{3}^{1}+\alpha_{3}^{2}-\alpha_{1}^{3}-\alpha_{2}^{3}\right)}$$

  - Yields

$$\begin{aligned}
LB_{32}^{TI} &= \min\left\{\frac{r_{32}-\ddot{Q}}{p_{3}},\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_{3}+3\widetilde{\widetilde{Q}}},\frac{r_{32}-\widehat{Q}}{p_{3}-3\widehat{Q}}\right\}\geq 0\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{32},(1-p_{3})/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{32},p_{3}/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{32}^{TI} &= \max\left\{\frac{r_{32}+\widetilde{Q}}{p_{3}},\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_{3}+3\widetilde{\widetilde{Q}}},\frac{r_{32}-\widehat{Q}}{p_{3}-3\widehat{Q}}\right\}\leq 1\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{32},(1-p_{3})/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{32},p_{3}/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{32}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_1^2,\beta_3^2=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^3,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_3+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_{3}+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_{3}+3\widetilde{\widetilde{Q}}\right)-3\left(r_{32}+\widetilde{\widetilde{Q}}\right)\right)\\
&= sgn\left(p_{3}-3r_{2}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{32}-\widehat{Q}}{p_{3}-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_{3}-3\widehat{Q}\right)+3\left(r_{32}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{32}-p_{3}\right)
\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{32}^{*}=\frac{r_{32}+\left(\beta_{2}^{3}-\beta_{1}^{2}\right)-\left(\alpha_{1}^{3}+\alpha_{2}^{3}\right)\left(1+\beta_{1}^{2}-\beta_{2}^{3}\right)}{p_{3}-3\left(\alpha_{1}^{3}+\alpha_{2}^{3}\right)}$$

    * Yields

$$\begin{aligned}
LB_{32}^{TI,u} &= \min\left\{\frac{r_{32}-\widetilde{Q}}{p_{3}},\frac{r_{32}-\widetilde{\widetilde{Q}}}{p_{3}-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widetilde{\widetilde{Q}}<\min\left\{r_{32},p_{3}/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{32}^{TI,u} &= \max\left\{\frac{r_{32}+\widetilde{Q}}{p_{3}},\frac{r_{32}-\widetilde{\widetilde{Q}}}{p_{3}-3\widetilde{\widetilde{Q}}}\right\}\leq 1\\
&\quad \widetilde{\widetilde{Q}}<\min\left\{r_{32},p_{3}/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{32}^{*}=\frac{r_{32}+\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)\left(\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)}{p_3+3\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)}$$

  - Yields

$$LB_{32}^{TIV} = \min\left\{\frac{r_{32}-\widetilde{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{32}}}{2},\sqrt{1-r_{32}},(1-p_3)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{32}^{TIV} = \max\left\{\frac{r_{32}+\widetilde{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{32}},p_3/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  - Proof:

    1. Evaluate $\partial LB_{32}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_3+3\widetilde{\widetilde{Q}}\right)-3\left(r_{32}+\widetilde{\widetilde{Q}}^2\right)\right)\\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{32}\right)\\ \Rightarrow sgn\left(\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{32}\right)<0\\ \Rightarrow\widetilde{\widetilde{Q}} &> 0\end{aligned}$$

    2. Minimize LB_{32}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{32}=0\\ \Rightarrow\ \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{32}}}{2}\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

  - Adding the uni-directional assumption

$$p_{32}^{*}=\frac{r_{32}-\left(\theta_1^2+\theta_1^3\right)-\left(\theta_1^3+\theta_2^3\right)\left(\theta_1^2-\theta_2^3\right)}{p_3-3\left(\theta_1^3+\theta_2^3\right)}$$

    * Yields

$$LB_{32}^{TIV,u} = \frac{r_{32}-\widetilde{Q}}{p_3} \qquad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{32}^{TIV,u} = \frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{32}},p_3/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

    * Proof: Same as above.

A.1.9 p^*_{33}

$$p^*_{33} = \frac{r_{33} + \overbrace{\left[\theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33} + \underset{\bullet}{\theta}{}^{31}_{33} + \underset{\bullet}{\theta}{}^{32}_{33}\right]}^{Q_{1,33}} - \overbrace{\left[\theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23} + \underset{\bullet}{\theta}{}^{33}_{31} + \underset{\bullet}{\theta}{}^{33}_{32}\right]}^{Q_{2,33}}}{\begin{array}{l} p_3 + \underbrace{\left[\underset{\bullet}{\theta}{}^{11}_{31} + \underset{\bullet}{\theta}{}^{12}_{31} + \underset{\bullet}{\theta}{}^{13}_{31} + \underset{\bullet}{\theta}{}^{21}_{31} + \underset{\bullet}{\theta}{}^{22}_{31} + \underset{\bullet}{\theta}{}^{23}_{31} + \underset{\bullet}{\theta}{}^{11}_{32} + \underset{\bullet}{\theta}{}^{12}_{32} + \underset{\bullet}{\theta}{}^{13}_{32} + \underset{\bullet}{\theta}{}^{21}_{32} + \underset{\bullet}{\theta}{}^{22}_{32} + \underset{\bullet}{\theta}{}^{23}_{32} + \theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33}\right]}_{Q_{3,3}} \\ - \underbrace{\left[\underset{\bullet}{\theta}{}^{31}_{11} + \underset{\bullet}{\theta}{}^{31}_{12} + \underset{\bullet}{\theta}{}^{31}_{13} + \underset{\bullet}{\theta}{}^{32}_{11} + \underset{\bullet}{\theta}{}^{32}_{12} + \underset{\bullet}{\theta}{}^{32}_{13} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \underset{\bullet}{\theta}{}^{31}_{21} + \underset{\bullet}{\theta}{}^{31}_{22} + \underset{\bullet}{\theta}{}^{31}_{23} + \underset{\bullet}{\theta}{}^{32}_{21} + \underset{\bullet}{\theta}{}^{32}_{22} + \underset{\bullet}{\theta}{}^{32}_{23} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q_{4,3}} \end{array}}$$

- $\underset{\bullet}{\theta}{}^{k'l'}_{kl}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{33} &= \frac{r_{33} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{33} &= \frac{r_{33} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{33} = \frac{r_{33} - \overbrace{\left[\theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23} + \underset{\bullet}{\theta}{}^{33}_{31} + \underset{\bullet}{\theta}{}^{33}_{32}\right]}^{Q^u_{2,33}}}{p_3 - \underbrace{\left[\underset{\bullet}{\theta}{}^{31}_{11} + \underset{\bullet}{\theta}{}^{32}_{11} + \underset{\bullet}{\theta}{}^{32}_{12} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \underset{\bullet}{\theta}{}^{31}_{21} + \underset{\bullet}{\theta}{}^{32}_{21} + \underset{\bullet}{\theta}{}^{32}_{22} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q^u_{4,3}}}$$

- Yields

$$\begin{aligned} LB^u_{33} &= \frac{r_{33} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB^u_{33} &= \frac{r_{33}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \min\{p_3, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{33}^{*}=\frac{r_{33}+\overbrace{\left[\alpha_3^1\beta_3^1+\alpha_3^1\beta_3^2+\alpha_3^1\beta_3^3+\alpha_3^2\beta_3^1+\alpha_3^2\beta_3^2+\alpha_3^2\beta_3^3+\alpha_3^3\beta_3^1+\alpha_3^3\beta_3^2\right]}^{Q_{1,33}}-\overbrace{\left[\alpha_1^3\beta_1^3+\alpha_1^3\beta_2^3+\alpha_1^3\beta_3^3+\alpha_2^3\beta_1^3+\alpha_2^3\beta_2^3+\alpha_2^3\beta_3^3+\alpha_3^3\beta_1^3+\alpha_3^3\beta_2^3\right]}^{Q_{2,33}}}{\begin{array}{c}p_3+\underbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^1\beta_2^1+\alpha_3^1\beta_2^2+\alpha_3^1\beta_2^3+\alpha_3^2\beta_2^1+\alpha_3^2\beta_2^2+\alpha_3^2\beta_2^3+\alpha_3^1\beta_3^1+\alpha_3^1\beta_3^2+\alpha_3^1\beta_3^3+\alpha_3^2\beta_3^1+\alpha_3^2\beta_3^2+\alpha_3^2\beta_3^3\right]}_{Q_{3,3}}\\-\underbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_2^2+\alpha_1^3\beta_3^2+\alpha_1^3\beta_1^3+\alpha_1^3\beta_2^3+\alpha_1^3\beta_3^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_2^2+\alpha_2^3\beta_3^2+\alpha_2^3\beta_1^3+\alpha_2^3\beta_2^3+\alpha_2^3\beta_3^3\right]}_{Q_{4,3}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,33} &= \left(\alpha_3^1+\alpha_3^2\right)+\left(\beta_3^1+\beta_3^2\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= 2\left(\theta_3^1+\theta_3^2\right)-\left(\theta_3^1+\theta_3^2\right)^2 && \text{(TIV)}\\
Q_{2,33} &= \left(\alpha_1^3+\alpha_2^3\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)+\left(\beta_1^3+\beta_2^3\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= 2\left(\theta_1^3+\theta_2^3\right)\left(1-\theta_3^1-\theta_3^2\right)+\left(\theta_1^3+\theta_2^3\right)^2 && \text{(TIV)}\\
Q_{3,3} &= 3\left(\alpha_3^1+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_3^1+\theta_3^2\right) && \text{(TIV)}\\
Q_{4,3} &= 3\left(\alpha_1^3+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^3+\theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{33}^{*}=\frac{r_{33}+\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)\left(1+\alpha_1^3+\alpha_2^3-\alpha_3^1-\alpha_3^2\right)}{p_2+3\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)}$$

  - Yields

$$LB_{33}^{TI} = \min\left\{\frac{r_{33}-\ddot{Q}}{p_3},\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{33},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{33}^{TI} = \max\left\{\frac{r_{33}+\widetilde{Q}}{p_3},\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{33},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{33}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_1^3,\beta_2^3=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_3+3\tilde{\tilde{Q}}\right)-3\left(r_{33}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_3-3r_{33}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_3-3\widehat{Q}\right)+3\left(r_{33}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{33}-p_3\right)\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{33}^{*}=\frac{r_{33}-\left(\alpha_1^3+\alpha_2^3\right)-\left(\beta_1^3+\beta_2^3\right)\left(1+\alpha_1^3+\alpha_2^3\right)}{p_2-3\left(\alpha_1^3+\alpha_2^3\right)}$$

    * Yields

$$LB_{33}^{TI,u} = \min\left\{\frac{r_{33}-\ddot{Q}}{p_3},\frac{r_{33}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\right\}\geq 0 \qquad \tilde{\tilde{Q}}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{33}^{TI,u} = \frac{r_{33}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\leq 1 \qquad \tilde{\tilde{Q}}=\begin{cases}0 & r_{33}<p_3/3\\ \min\left\{r_{33},p_3/3,\ddot{Q}\right\} & \text{otherwise}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{33}^{*}=\frac{r_{33}+2\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)-\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)^2}{p_2+3\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)}$$

  - Yields

$$\begin{aligned}
LB_{33}^{TIV} &= \min\left\{\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}},\frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\min\left\{(1-p_3)/3,\widetilde{Q}\right\},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}\geq 2p_3/3\\ \min\left\{\frac{(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_3/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{33}^{TIV} &= \min\left\{\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}},\frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\begin{cases}0 & r_{33}\geq 2p_3/3\\ \min\left\{\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(1-p_3)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}<2p_3/3\\ \min\left\{\frac{(2/3)p_3-\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_3/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$
\begin{aligned}
sgn \left(\frac{\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= sgn \left(\left(-2 - 2\tilde{\tilde{Q}} \right) \left(p_3 - 3\tilde{\tilde{Q}} \right) + 3 \left(r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2 \right) \right) \\
&= sgn \left(-(2/3)p_3 \left(1 + \tilde{\tilde{Q}} \right) + \tilde{\tilde{Q}}^2 + r_{33} \right) \\
\Rightarrow sgn \left(\frac{\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) \Bigg|_{\tilde{\tilde{Q}}=0} &= sgn \left(-(2/3)p_3 + r_{33} \right) \gtrless 0 \\
\Rightarrow sgn \left(\frac{\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) \Bigg|_{\tilde{\tilde{Q}}=1} &= sgn \left(-(4/3)p_3 + 1 + r_{33} \right) \gtrless 0
\end{aligned}
$$

2. Ensure $r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2 \geq 0$

$$
\begin{aligned}
& r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2 \geq 0 \\
\Rightarrow \; & \tilde{\tilde{Q}}^2 + 2\tilde{\tilde{Q}} - r_{33} \leq 0 \\
\Rightarrow \; & \tilde{\tilde{Q}} \leq \frac{-2 + \sqrt{4 + 4r_{33}}}{2} \\
\Rightarrow \; & \tilde{\tilde{Q}} \leq -1 + \sqrt{1 + r_{33}}
\end{aligned}
$$

3. Minimize $\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{33} < 2p_3/3$

$$
\begin{aligned}
\frac{\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \quad & \propto \quad -(2/3)p_3 \left(1 + \tilde{\tilde{Q}} \right) + \tilde{\tilde{Q}}^2 + r_{33} = 0 \\
\Rightarrow \quad & \tilde{\tilde{Q}}^* = \frac{(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}
\end{aligned}
$$

4. Maximize $\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{33} > 2p_3/3$

$$
\begin{aligned}
\frac{\partial \left(\frac{r_{33} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_3 - 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \quad & \propto \quad -(2/3)p_3 \left(1 + \tilde{\tilde{Q}} \right) + \tilde{\tilde{Q}}^2 + r_{33} = 0 \\
\Rightarrow \quad & \tilde{\tilde{Q}}^* = \frac{(2/3)p_3 - \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}
\end{aligned}
$$

Note: If $\sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]} = .$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_3+3\widehat{Q}\right)-3\left(r_{33}+2\widehat{Q}-\widehat{Q}^2\right)\right)\\
&= sgn\left((2/3)p_3\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{33}\right)\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_3-r_{33}\right)\gtrless 0\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{33}\right)<0
\end{aligned}$$

6. Maximize $\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{33}<2p_3/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}} &\propto (2/3)p_3\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{33}=0\\
&\Rightarrow \widehat{Q}^*=\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\Rightarrow\widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}<\frac{r_{33}}{p_3}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p^*_{33} = \frac{r_{33} - 2\left(\theta_1^3 + \theta_2^3\right) - \left(\theta_1^3 + \theta_2^3\right)^2}{p_2 - 3\left(\theta_1^3 + \theta_2^3\right)}$$

∗ Yields

$$LB_{33}^{TIV} = \frac{r_{33} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_3 - 3\widetilde{\widetilde{Q}}} \geq 0$$

$$\widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{33} \geq 2p_3/3 \\ \min\left\{\frac{(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}, (-1 + \sqrt{1 + r_{33}}), p_3/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases},$$

$$\widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{33}^{TIV} = \frac{r_{33} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_3 - 3\widetilde{\widetilde{Q}}} \geq 0$$

$$\widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{33} < 2p_3/3 \\ \min\left\{\frac{(2/3)p_3 - \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}, (-1 + \sqrt{1 + r_{33}}), p_3/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases},$$

$$\widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

∗ Proof: Same as above.

A.2 Tightening the Bounds

A.2.1 Shape Restrictions

$p^*_{11}, p^*_{22}, p^*_{33}$

$$\begin{aligned} LB^S_{kk} &= \max\left\{\sup_{k'\neq k} LB_{kk'}, \sup_{k'\neq k} LB_{k'k}\right\} \\ UB^S_{kk} &= UB_{kk} \end{aligned}$$

$p^*_{12}, p^*_{13}, p^*_{23}$

$$\begin{aligned} LB^S_{kk} &= LB_{kk} \\ UB^S_{kk} &= \min\{UB_{11}, UB_{12}, UB_{22}\} \end{aligned}$$

$p^*_{21}, p^*_{31}, p^*_{32}$

$$\begin{aligned} LB^S_{kk} &= LB_{kk} \\ UB^S_{kk} &= \min\{UB_{11}, UB_{21}, UB_{22}\} \end{aligned}$$

A.2.2 Level Set Restrictions

$$p^*_{kl}(x) = \frac{r_{kl}(x) + Q_{1,kl}(x) - Q_{2,kl}(x)}{p_k(x) + Q_{3,k}(x) - Q_{4,k}(x)}$$

- Let $Q(x)$ be probability of misclassification conditional on $X = x$. Then

$$\sum_x p_x Q(x) \leq Q$$

- Implies

$$Q(x) = \begin{cases} Q/p_x & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

- Bounds
  - Bounds on $p^*_{kl}(x)$ are identical to baseline with Q replaced by $Q(x)$
  - After bounding $P^*_{01}(x)$, impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired

A.2.3 Monotonicity Restrictions

$$p^*_{kl}(u) = \frac{r_{kl}(u) + Q_{1,kl}(u) - Q_{2,kl}(u)}{p_k(u) + Q_{3,k}(u) - Q_{4,k}(u)}$$

- Let $Q(u)$ be probability of misclassification conditional on $U = u$. Then

$$\sum_u p_u Q(u) \leq Q$$

- Implies

$$Q(u) = \begin{cases} Q/p_u & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

- Bounds
  - Bounds on $p^*_{kl}(u)$ are identical to baseline with Q replaced by $Q(u)$
  - After bounding $P^*_{01}(u)$, impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired
- Adding level set restrictions

$$p^*_{kl}(x,u) = \frac{r_{kl}(x,u) + Q_{1,kl}(x,u) - Q_{2,kl}(x,u)}{p_k(x,u) + Q_{3,k}(x,u) - Q_{4,k}(x,u)}$$

- Let $Q(x,u)$ be probability of misclassification conditional on $X = x, U = u$. Then

$$\sum_x p_{xu} Q(x,u) \leq Q(u)$$

- Implies

$$Q(x,u) = \begin{cases} Q/\left(p_{xu} p_u\right) & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

  where $p_{xu} = \Pr(X = x | U = u)$
- Bounds
  - Bounds on $p^*_{kl}(x,u)$ are identical to baseline with Q replaced by $Q(x,u)$
  - After bounding $P^*_{01}(x,u)$, impose shape if desired
  - Derive bounds on $P^*_{01}(u)$
  - Impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired

B Supplemental Figures: Intragenerational Dynamics

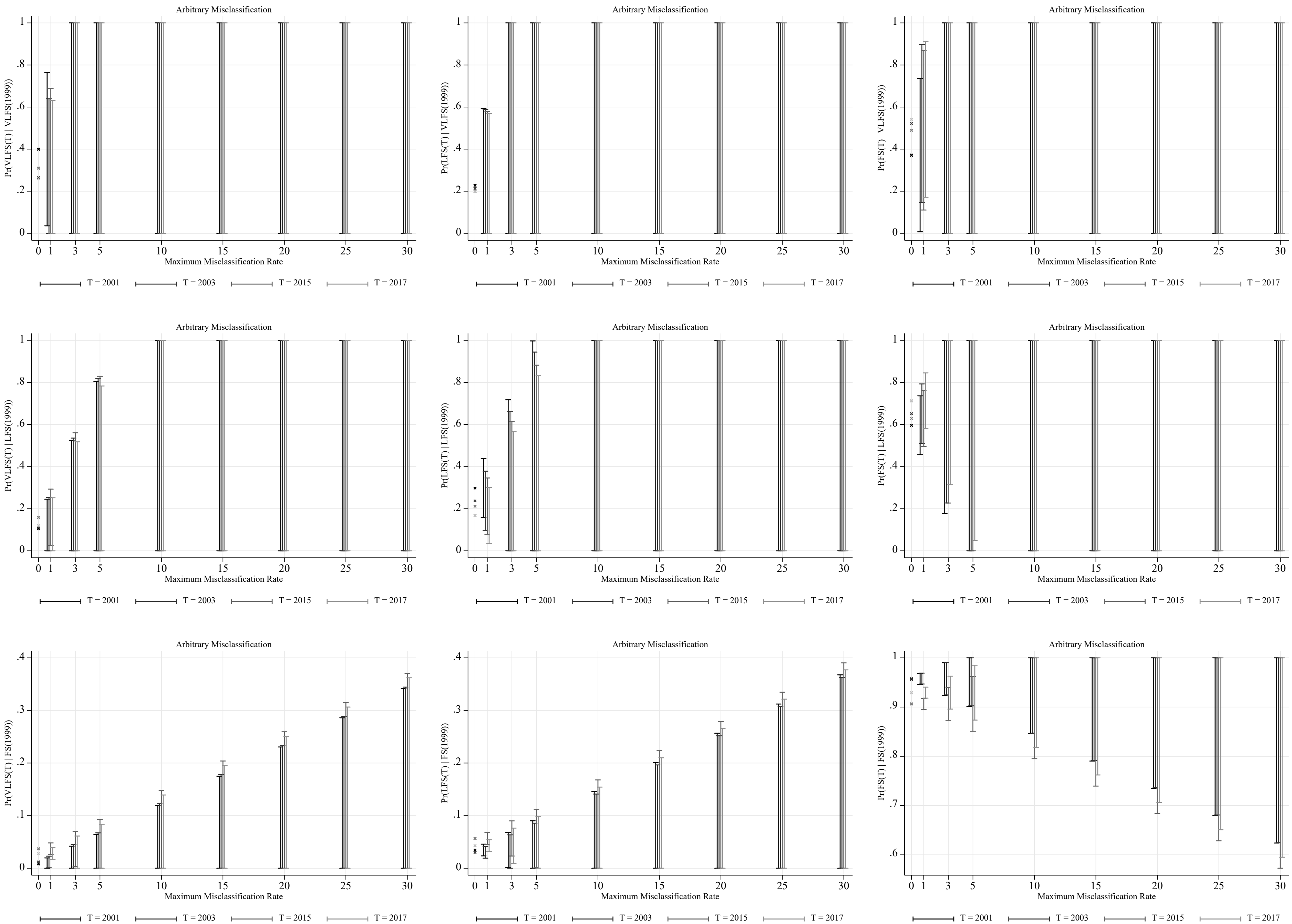

Figure B1. Bounds on Transition Probabilities by Time Period: Arbitrary Errors.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

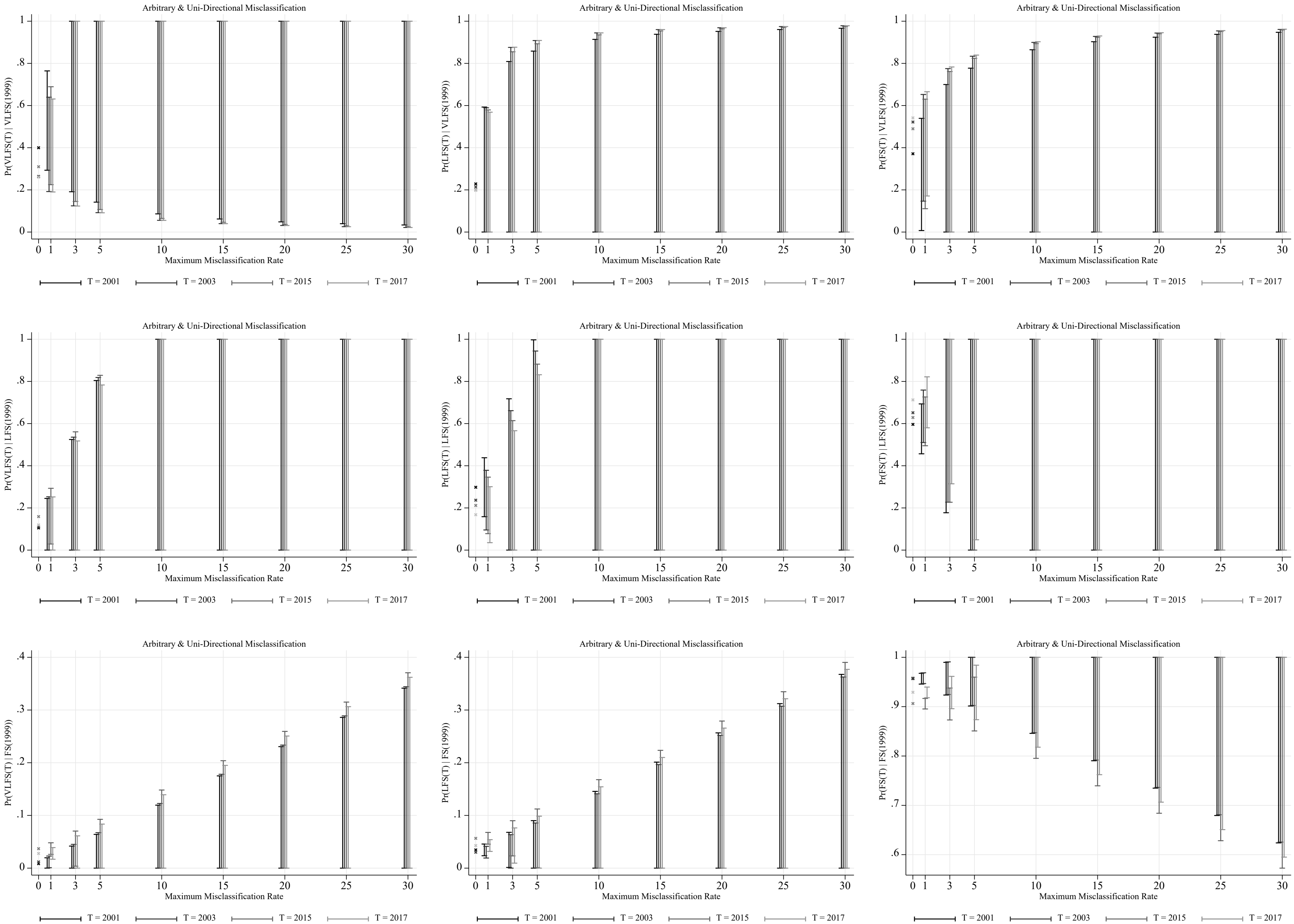

Figure B2. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

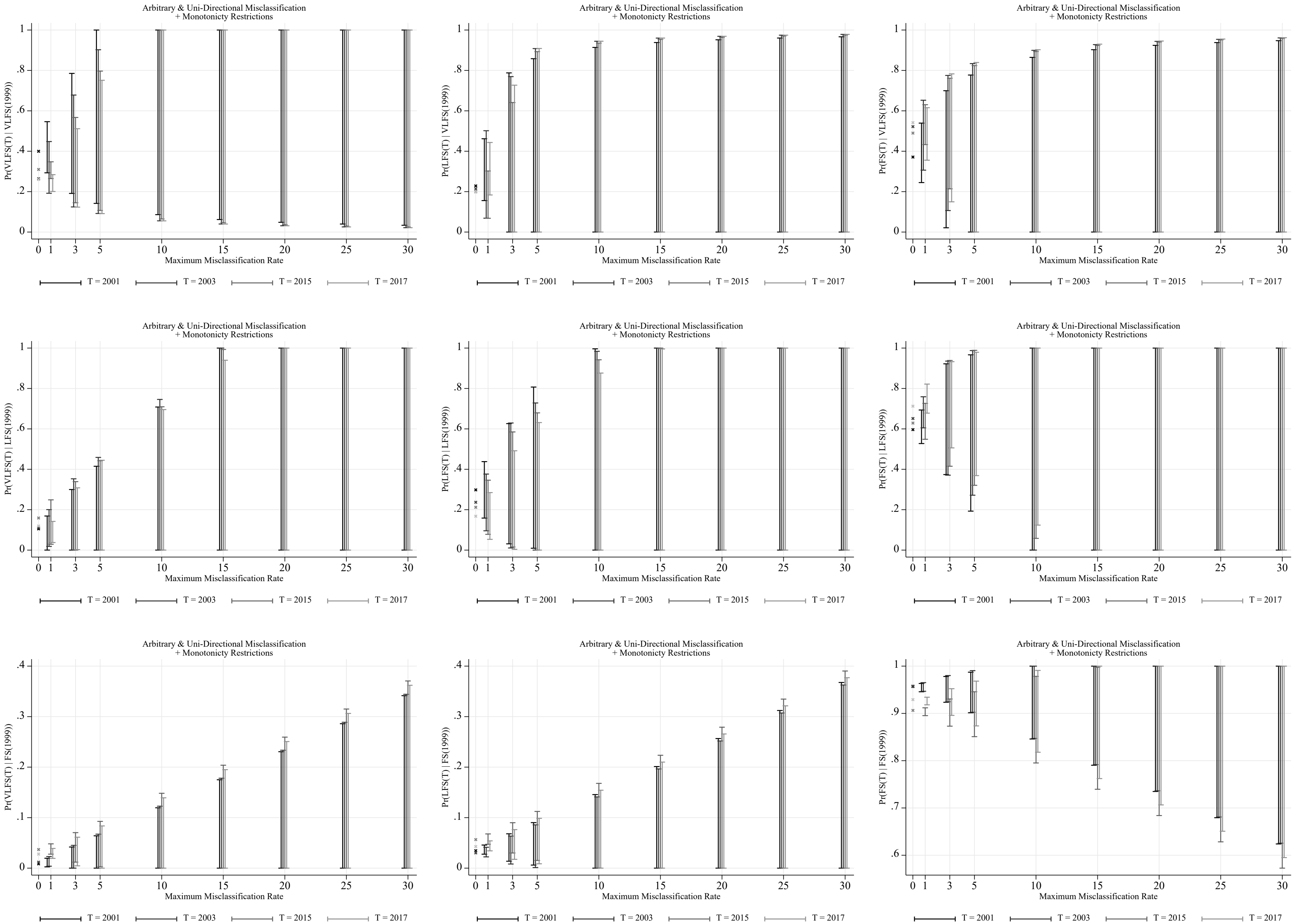

Figure B3. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

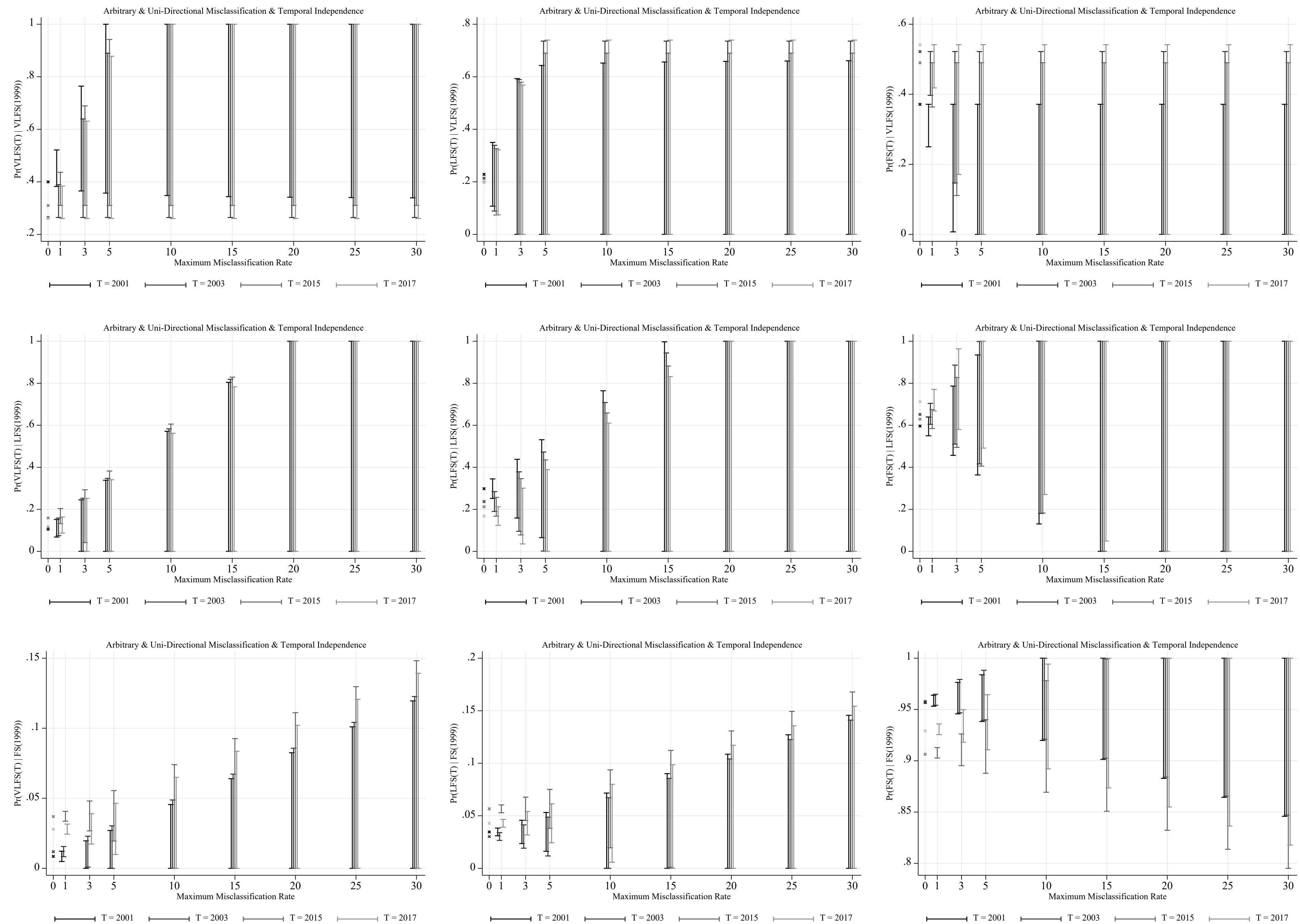

Figure B4. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

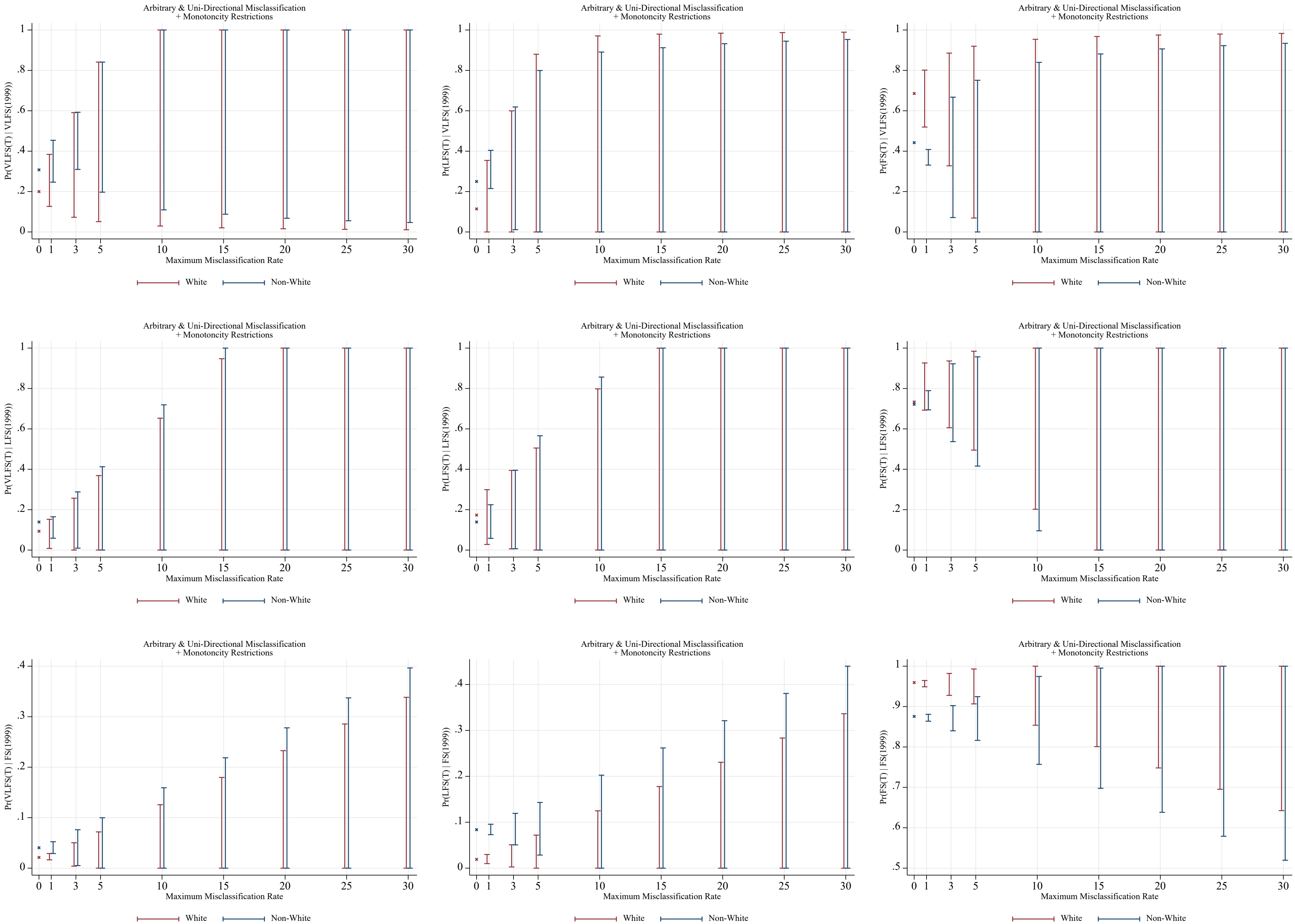

Figure B5. Bounds on Transition Probabilities from 1999 to 2017 by Race: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Race refers to the head of the household. See text for more details.

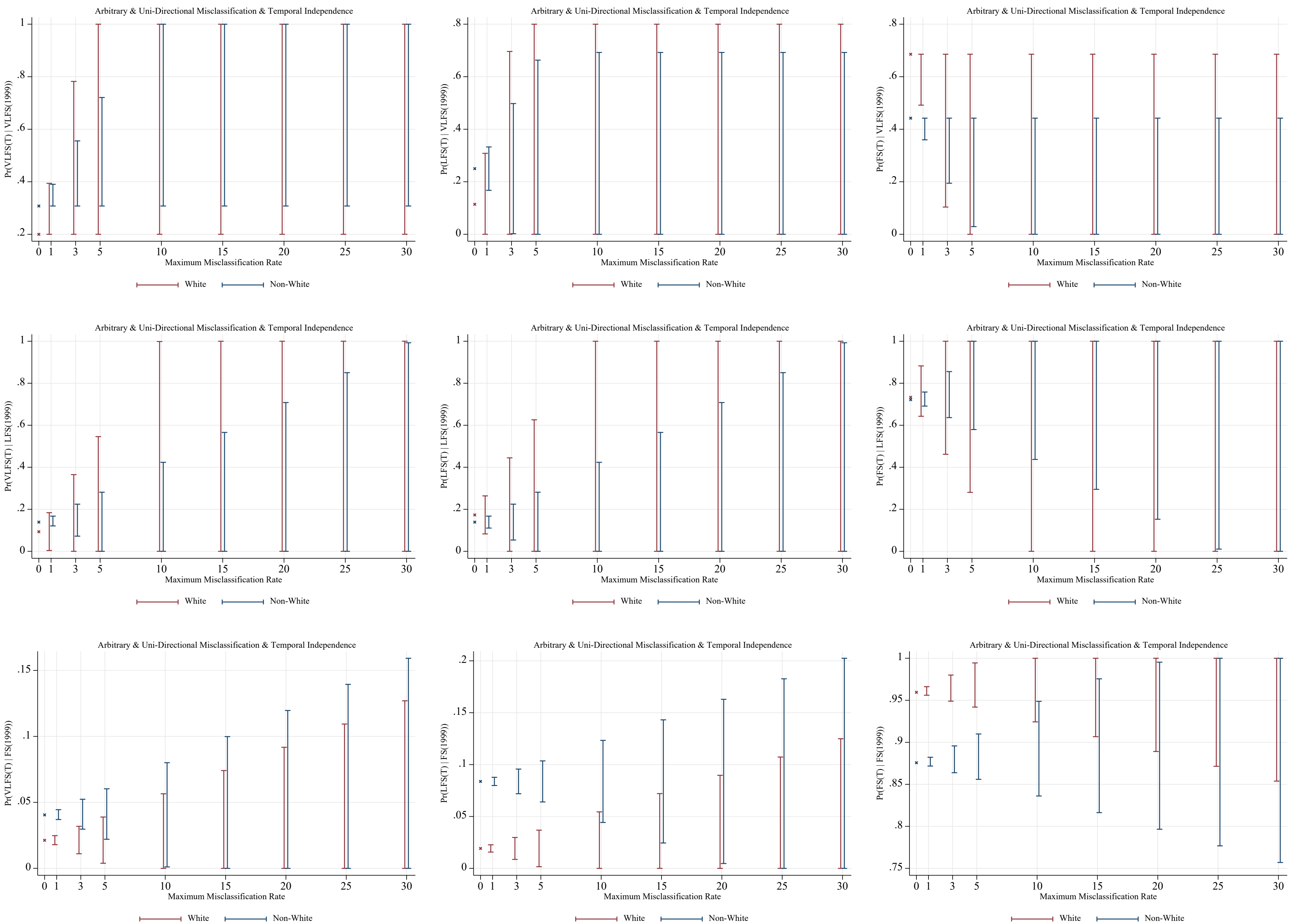

Figure B6. Bounds on Transition Probabilities from 1999 to 2017 by Race: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Race refers to the head of the household. See text for more details.

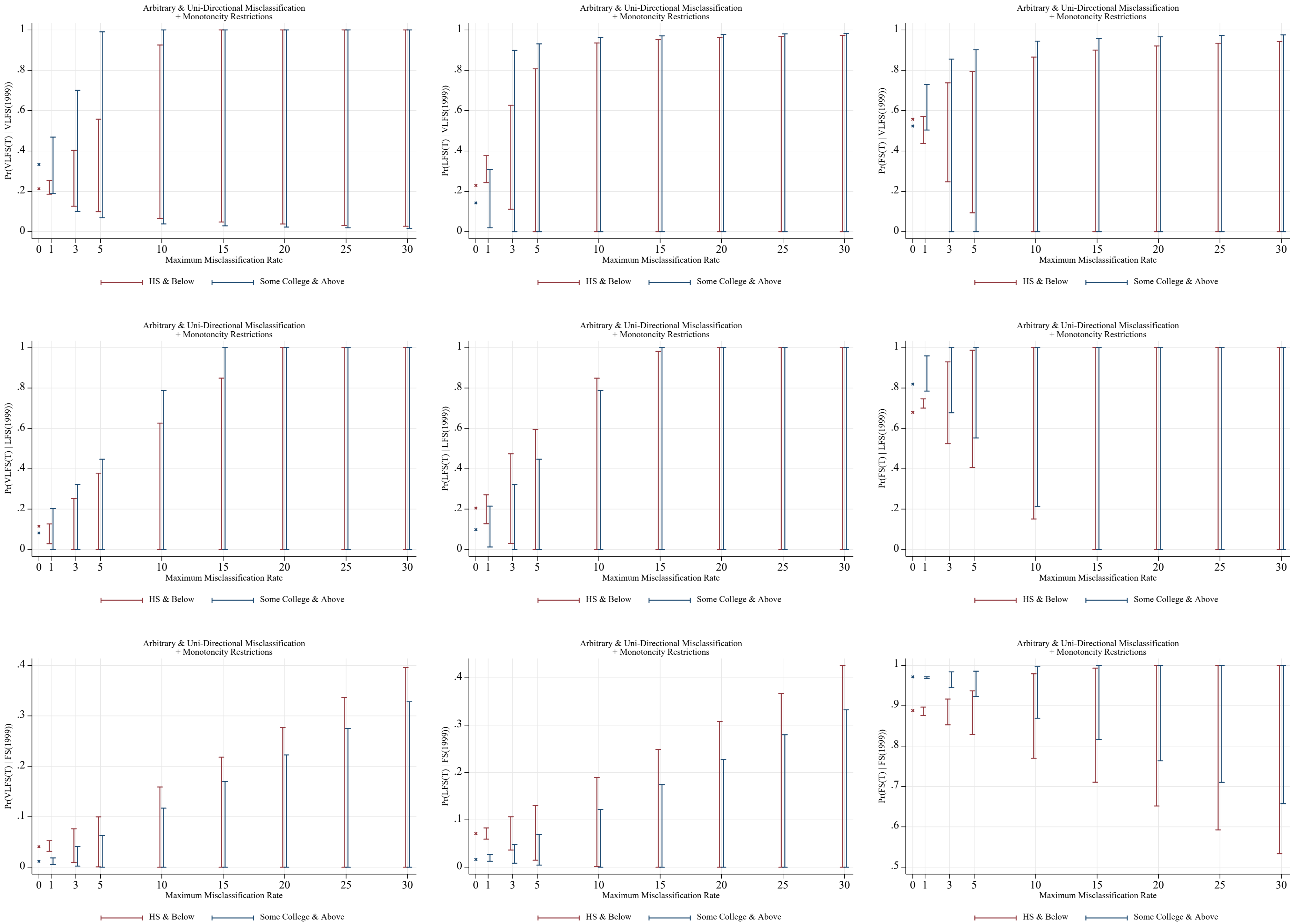

Figure B7. Bounds on Transition Probabilities from 1999 to 2017 by Education: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Education refers to the head of the household. See text for more details.

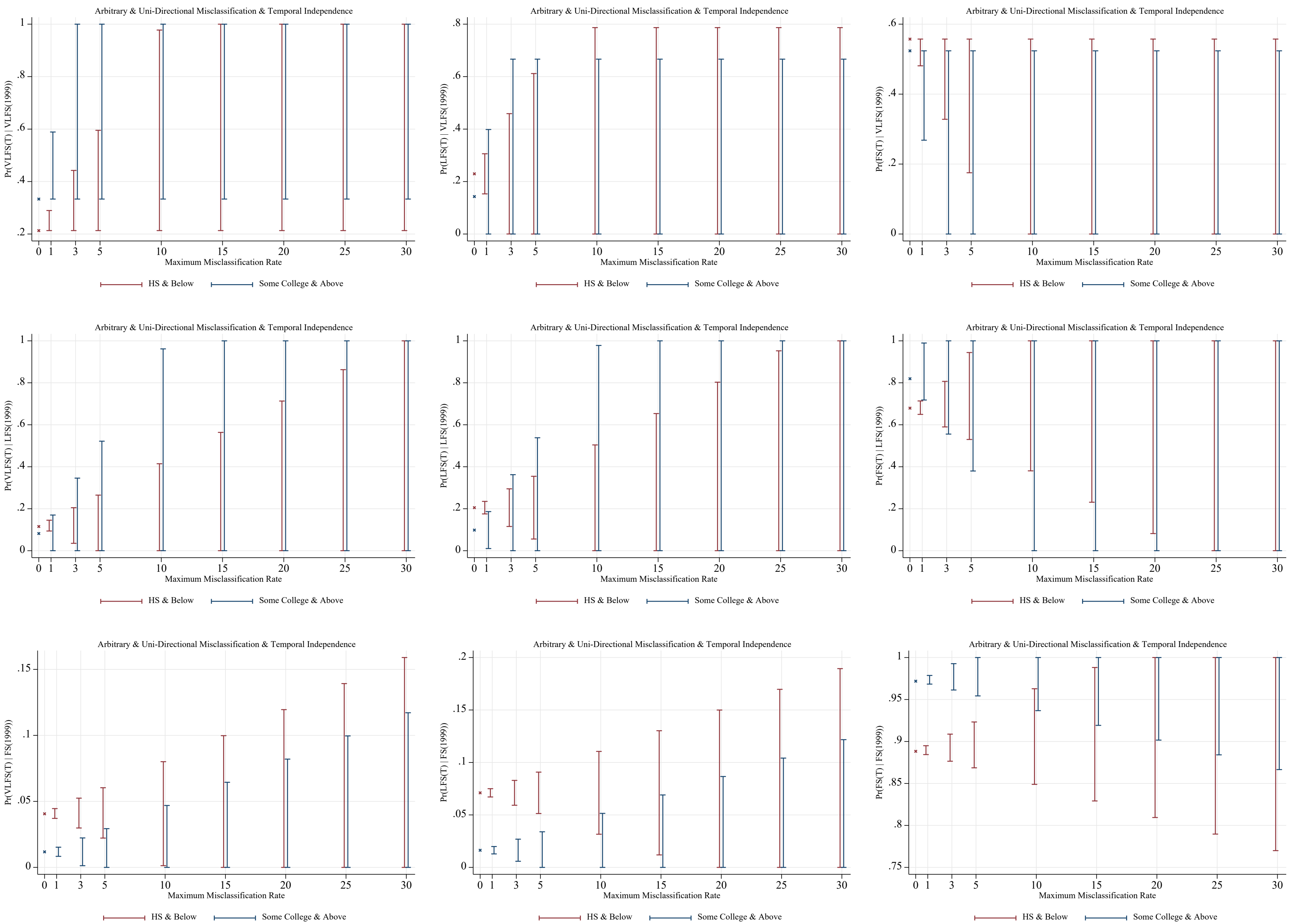

Figure B8. Bounds on Transition Probabilities from 1999 to 2017 by Education: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Education refers to the head of the household. See text for more details.

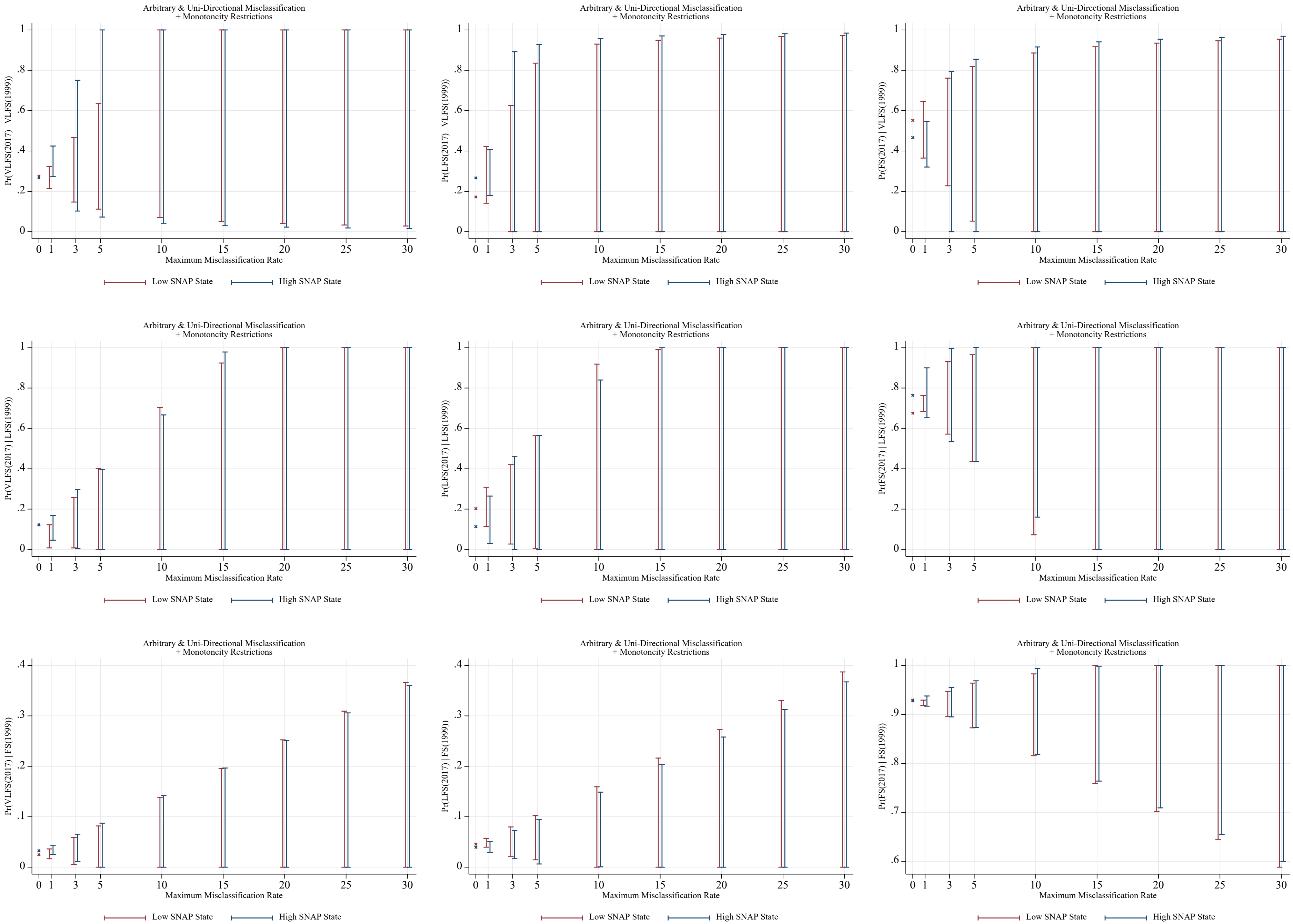

Figure B9. Bounds on Transition Probabilities from 1999 to 2017 by State SNAP Participation Rate: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. High (Low) SNAP State has a SNAP participation rate among the eligible population above (below) the median. See text for more details.

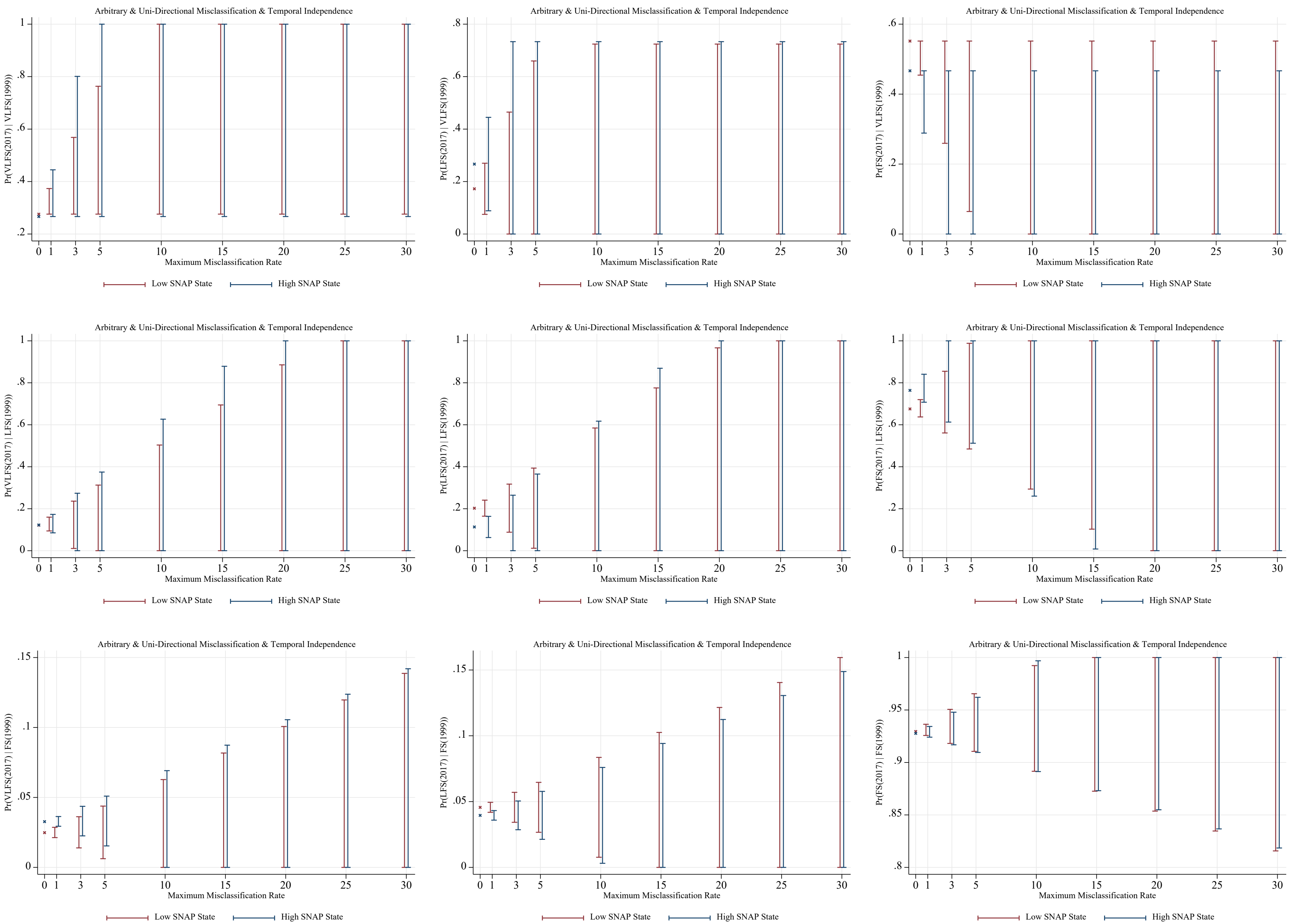

Figure B10. Bounds on Transition Probabilities from 1999 to 2017 by State SNAP Participation Rate: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. High (Low) SNAP State has a SNAP participation rate among the eligible population above (below) the median. See text for more details.

C Supplemental Figures: Intergenerational Dynamics

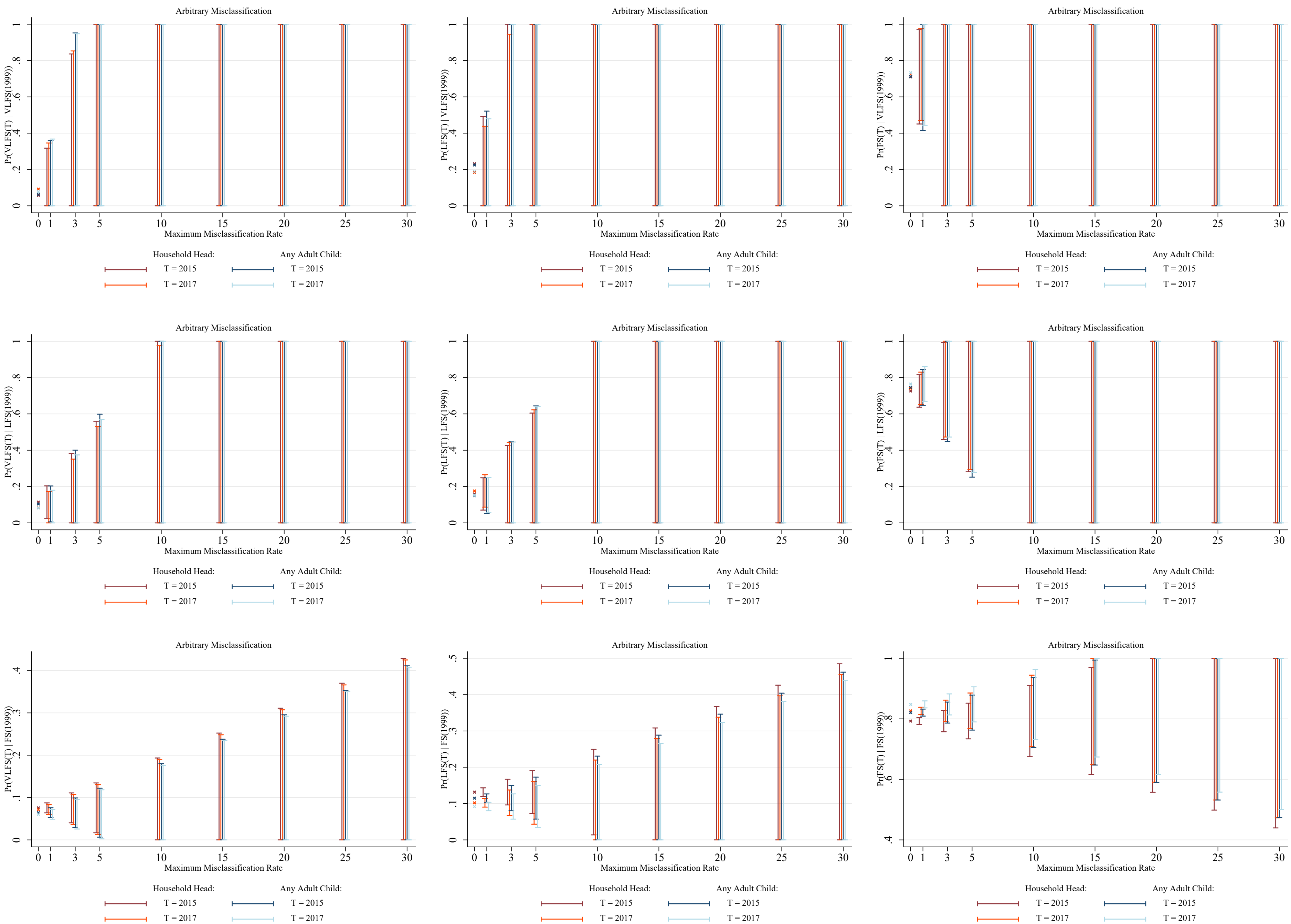

Figure C1. Bounds on Transition Probabilities by Time Period: Arbitrary Errors.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

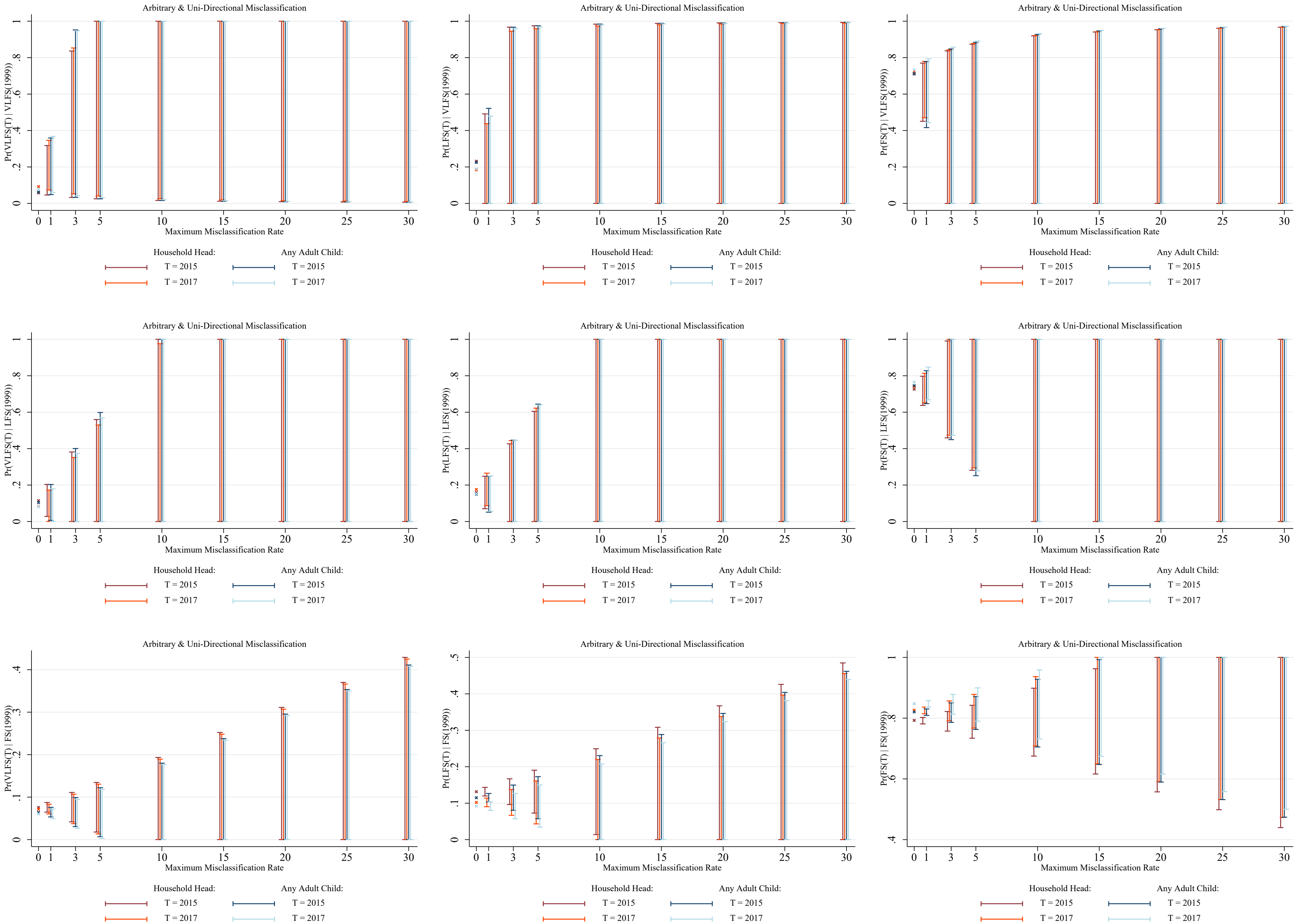

Figure C2. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

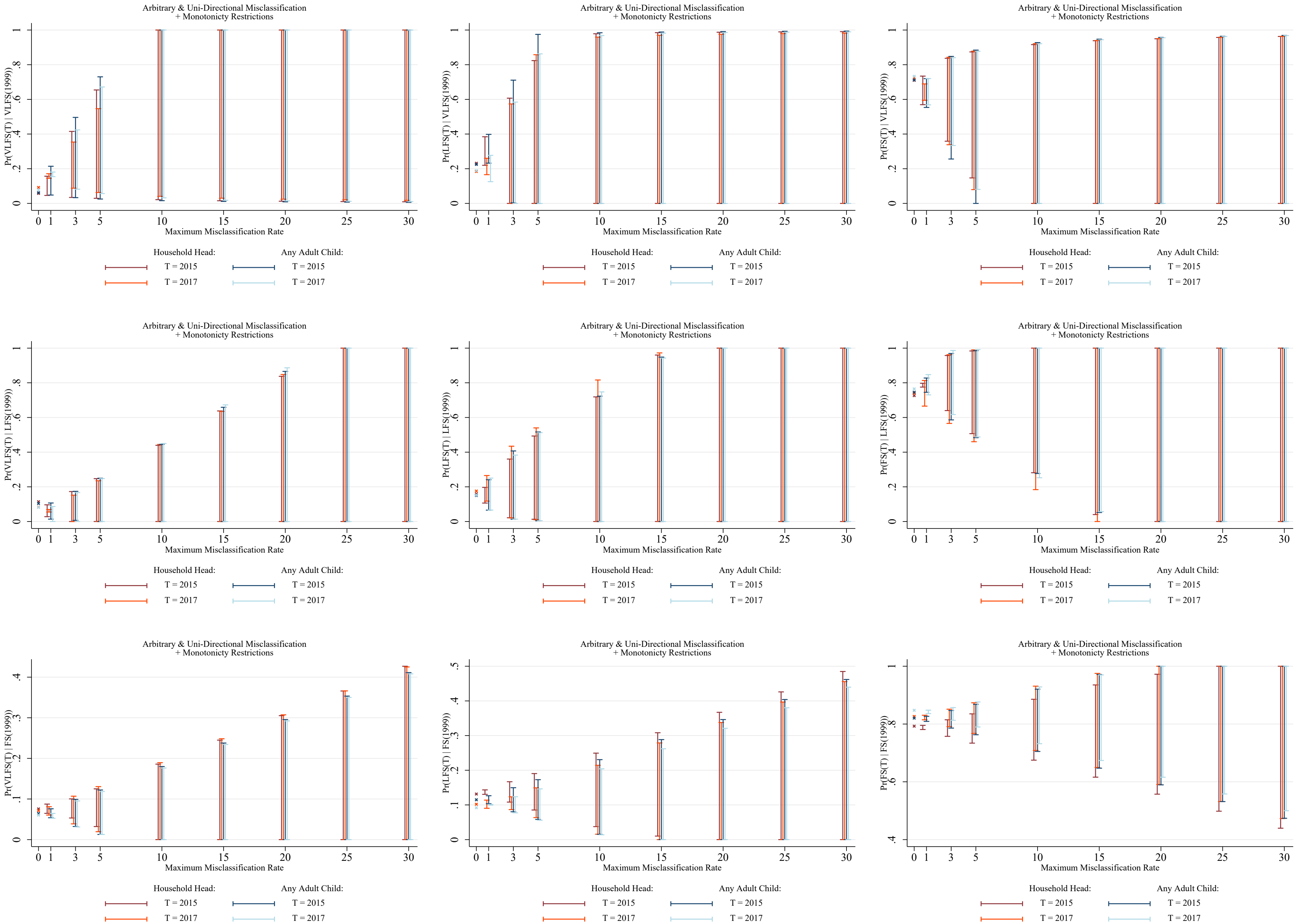

Figure C3. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

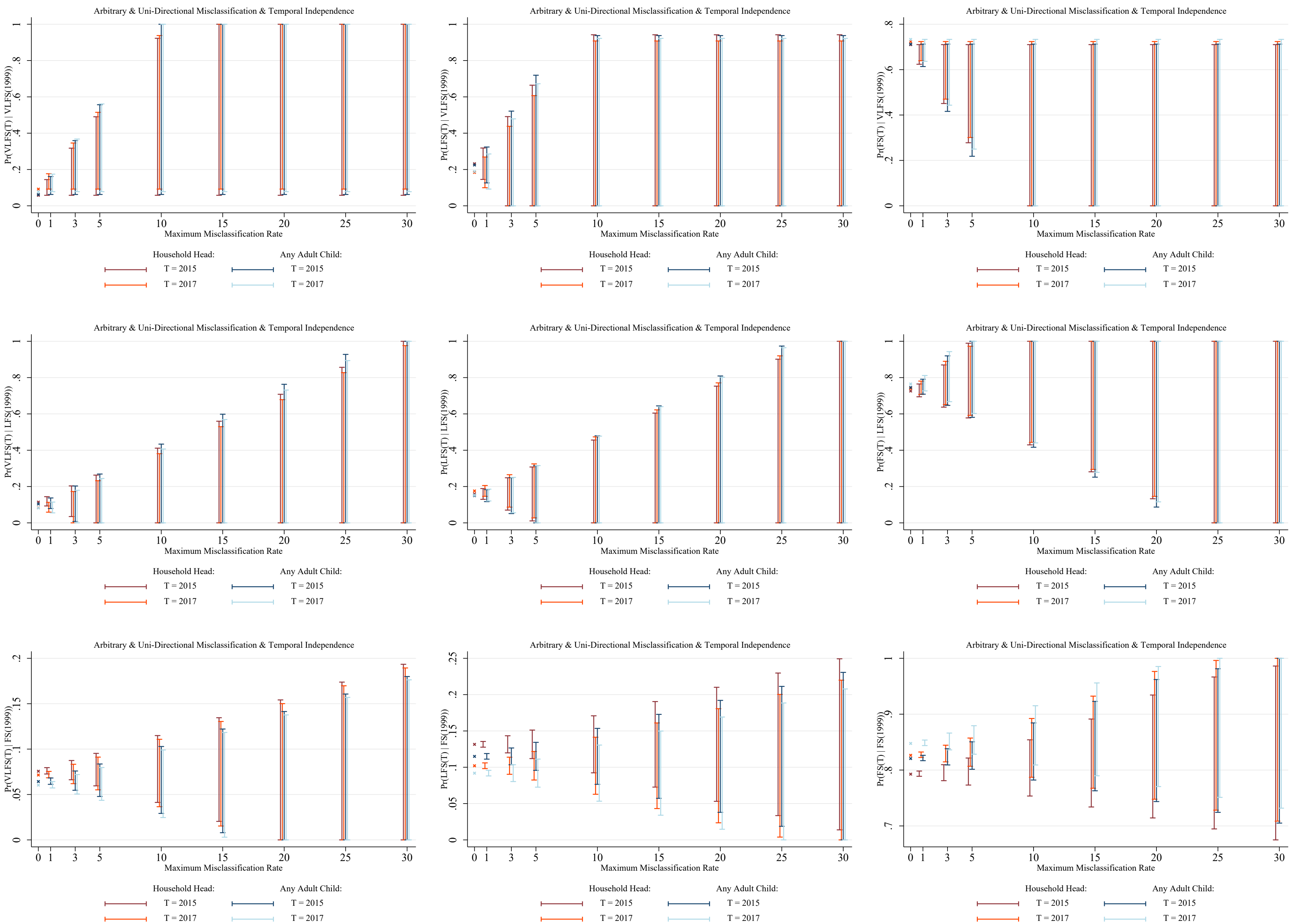

Figure C4. Bounds on Transition Probabilities by Time Period: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. T denotes the terminal time period used to compute the transition probabilities. Initial period is 1999. See text for more details.

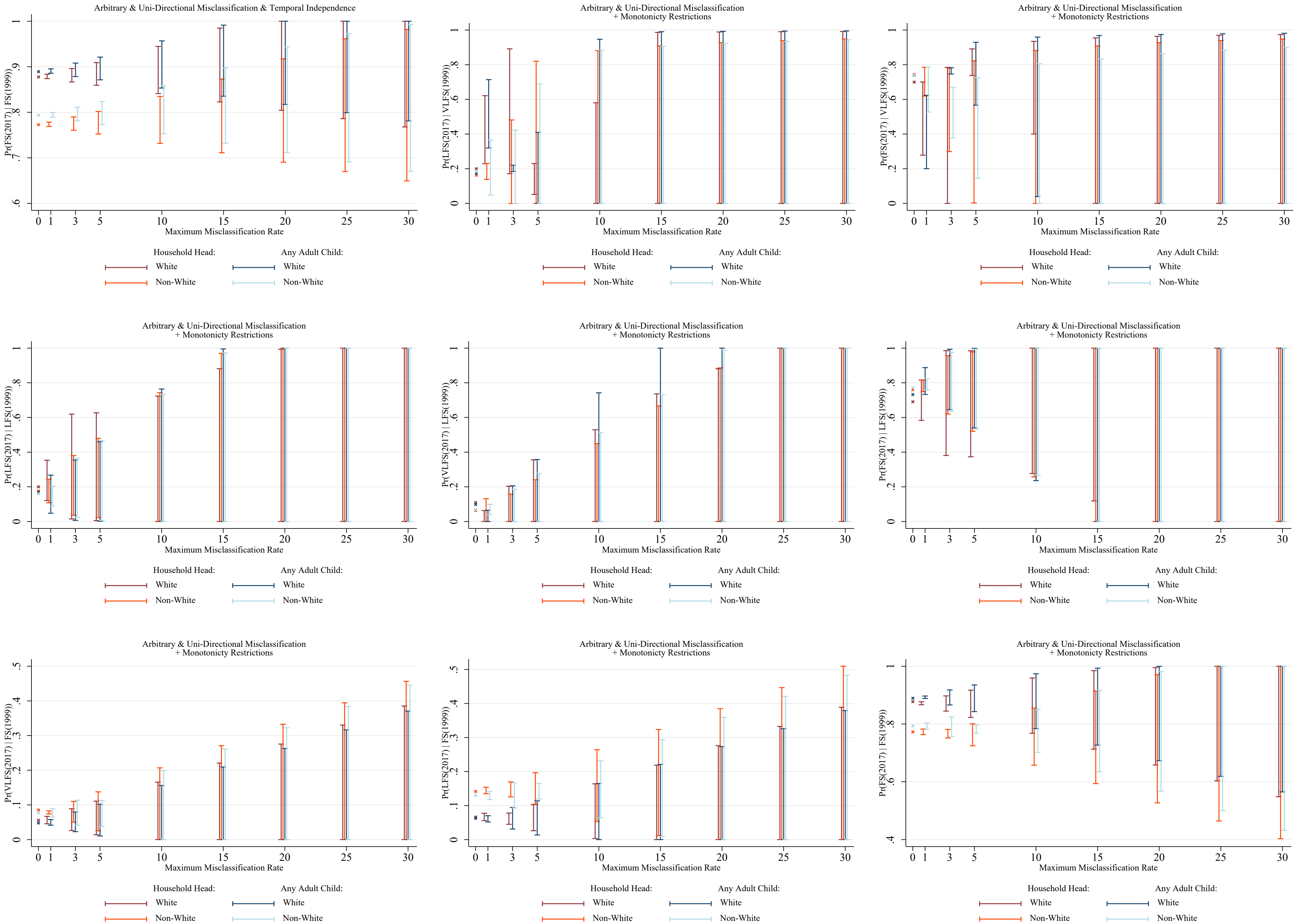

Figure C5. Bounds on Transition Probabilities from 1999 to 2015 by Race: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Race refers to the head of the household. See text for more details.

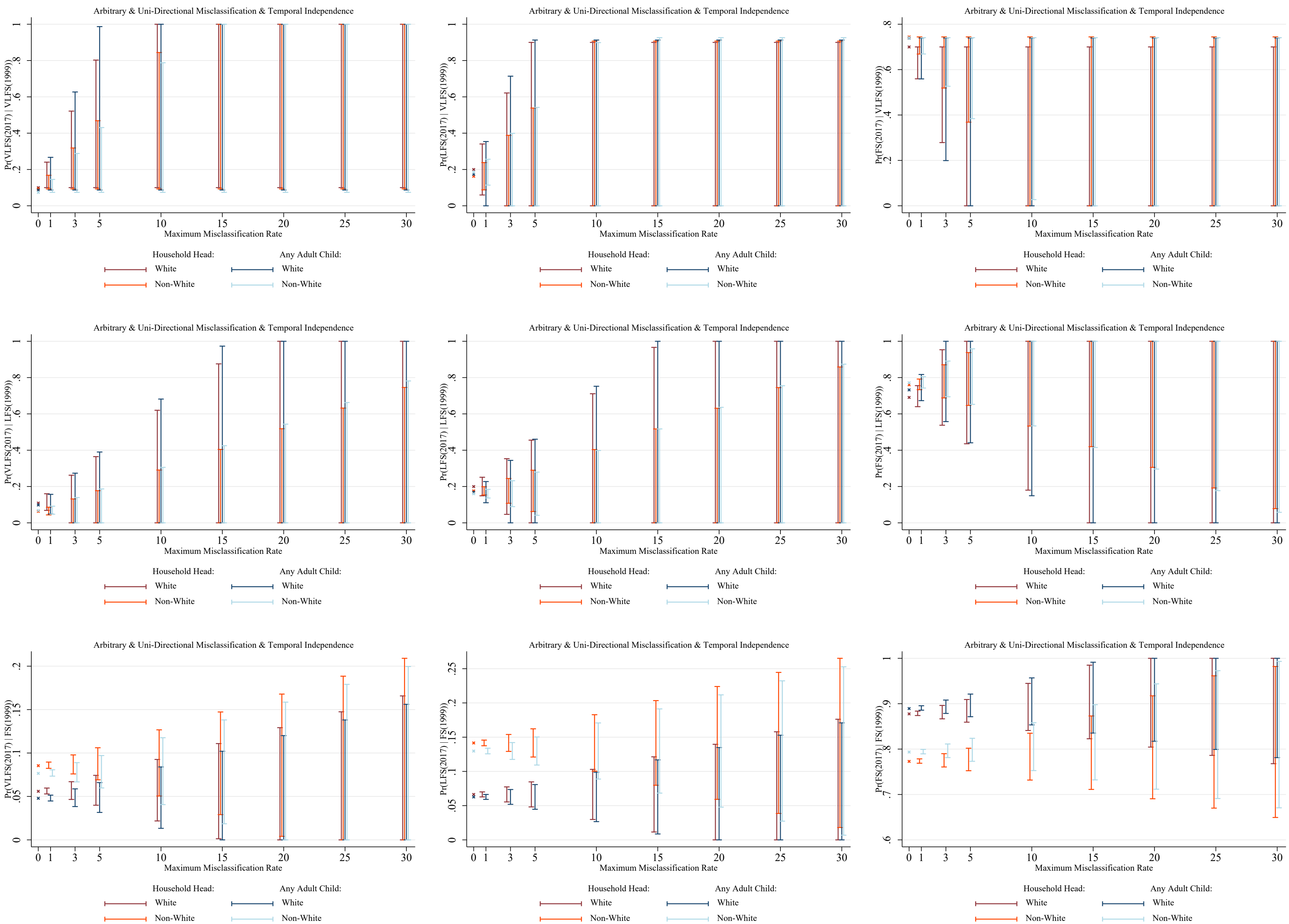

Figure C6. Bounds on Transition Probabilities from 1999 to 2015 by Race: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Race refers to the head of the household. See text for more details.

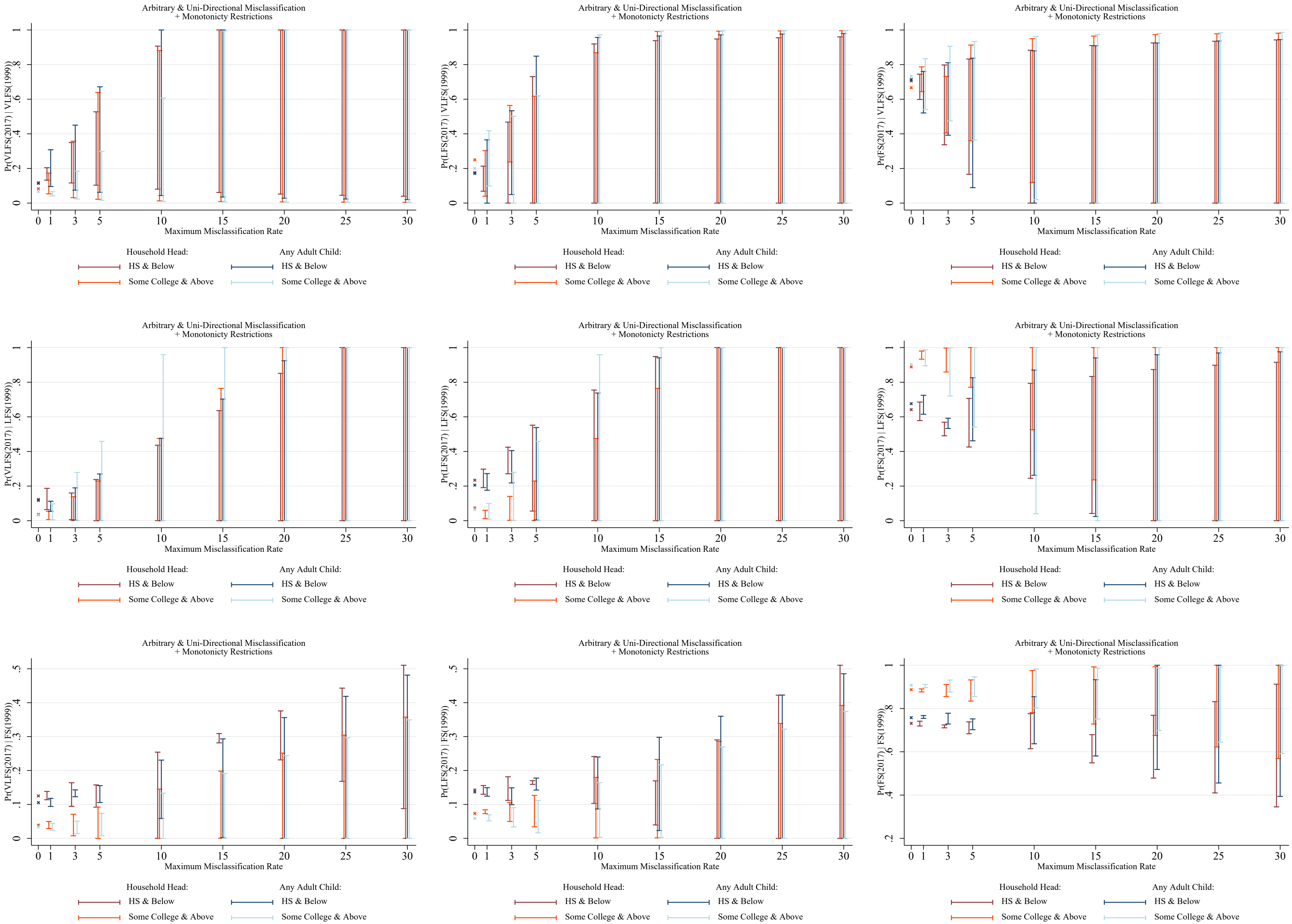

Figure C7. Bounds on Transition Probabilities from 1999 to 2015 by Education: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Education refers to the head of the household. See text for more details.

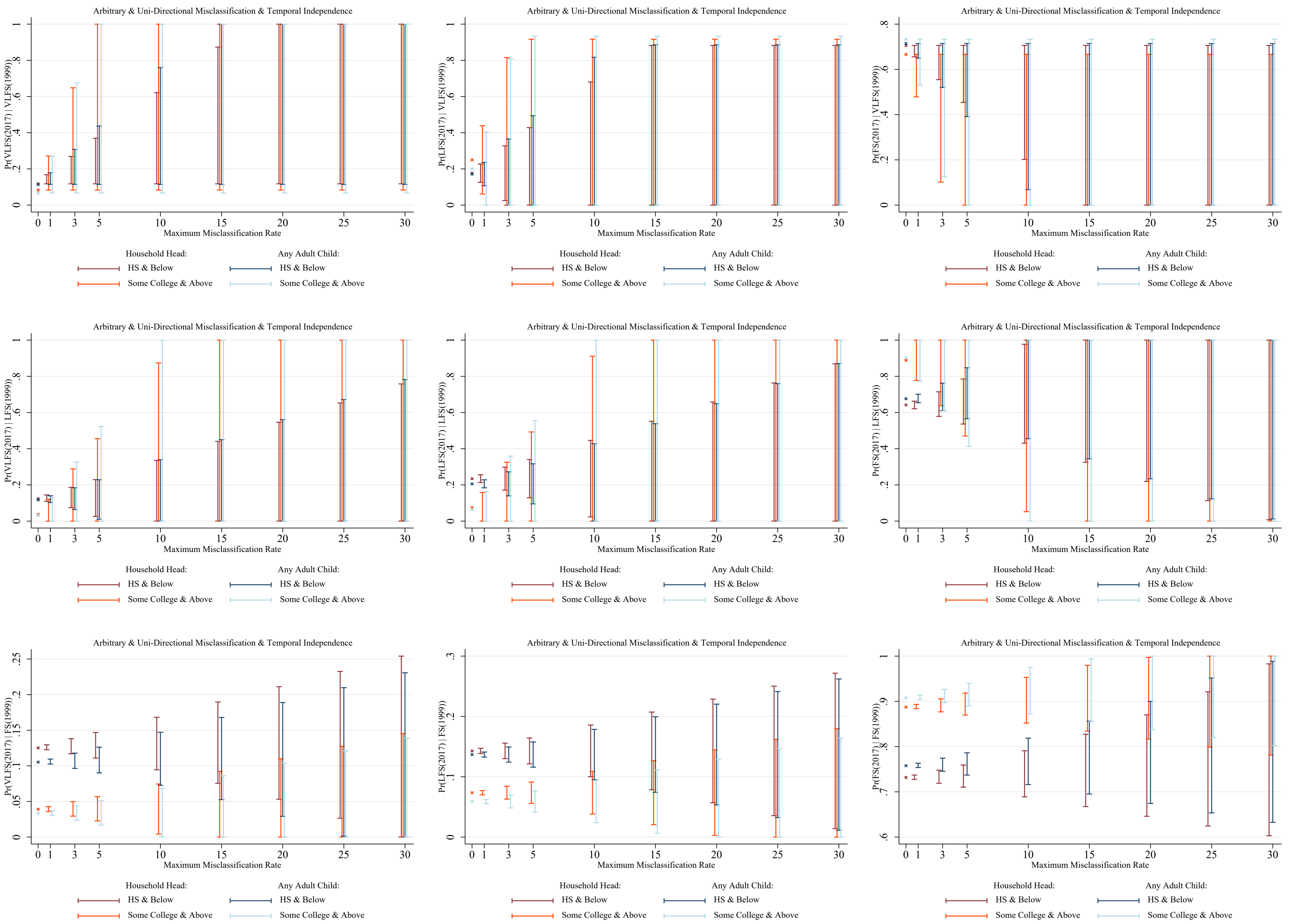

Figure C8. Bounds on Transition Probabilities from 1999 to 2015 by Education: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.
Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. Education refers to the head of the household. See text for more details.

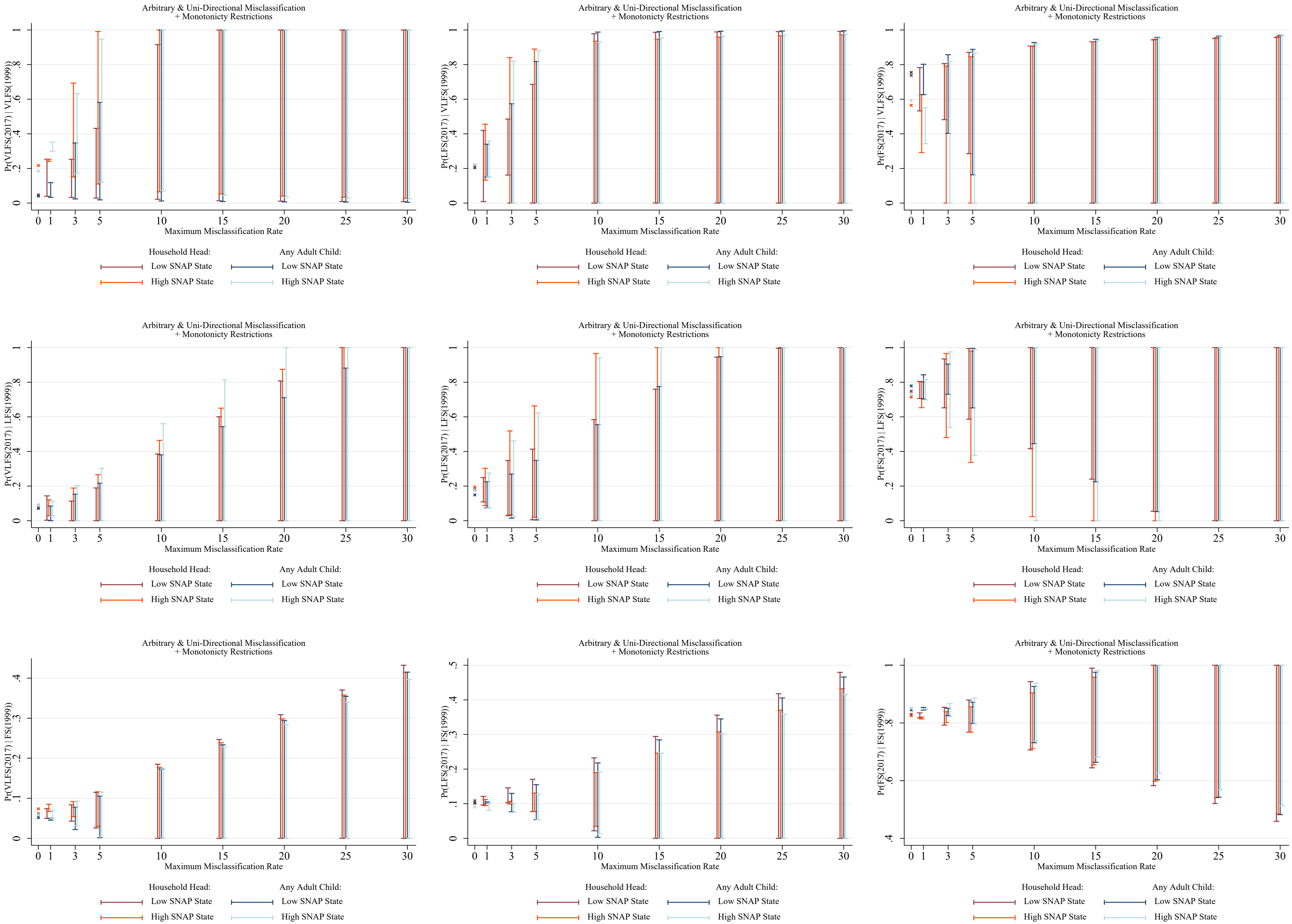

Figure C9. Bounds on Transition Probabilities from 1999 to 2015 by State SNAP Participation Rate: Arbitrary Errors + Uni-Directional Misclassification + Monotonicity Restrictions.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. High (Low) SNAP State has a SNAP participation rate among the eligible population above (below) the median. See text for more details.

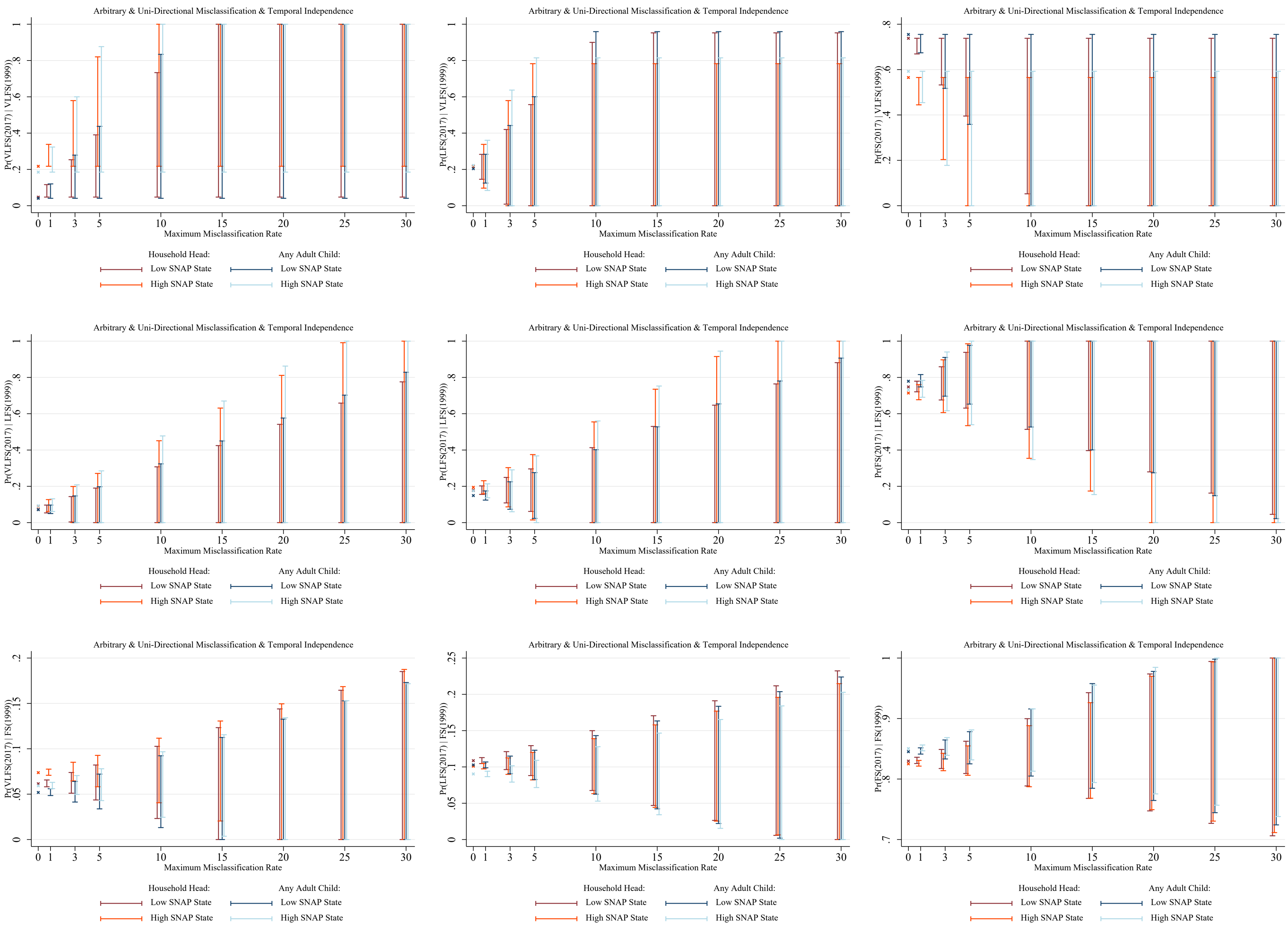

Figure C10. Bounds on Transition Probabilities from 1999 to 2015 by State SNAP Participation Rate: Arbitrary Errors + Uni-Directional Misclassification + Temporal Independence.

Notes: VFLS = Very Low Food Security. LFS = Low Food Security. FS = Food Secure. High (Low) SNAP State has a SNAP participation rate among the eligible population above (below) the median. See text for more details.